

A cell filtration of mixed tensor space

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Abstract

We construct a cellular basis of the walled Brauer algebra which has similar properties as the Murphy basis of the group algebra of the symmetric group. In particular, the restriction of a cell module to a certain subalgebra can be easily described via this basis. Furthermore, the mixed tensor space possesses a filtration by cell modules – although not by cell modules of the walled Brauer algebra itself, but by cell modules of its image in the algebra of endomorphisms of mixed tensor space.

Key words: walled Brauer algebra, mixed tensor space, cellular algebra, filtration, annihilator

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1. Introduction

Throughout, let n be a natural number, r, s and m non-negative integers and let R be a commutative ring. For $x \in R$, the walled Brauer algebra $B_{r,s}(x)$ is the subalgebra of the Brauer algebra $B_{r+s}(x)$ spanned by certain diagrams, the walled Brauer diagrams. Let V be a free R -module of rank n and let $V^* = \text{Hom}_R(V, R)$. Then the walled Brauer algebra for the parameter $x = n$ acts on the mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$ and satisfies Schur-Weyl duality together with the action of the universal enveloping algebra U of the general linear algebra (see [1, 13, 20, 4, 5]).

This situation is very similar to (in fact a generalization of) the classical Schur-Weyl duality where both the group algebra $R\mathfrak{S}_m$ of the symmetric group and U act on the ordinary tensor space $V^{\otimes m}$ ([18, 21, 9]). The group

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algebra of the symmetric group is a cellular algebra in the sense of Graham and Lehrer ([8]) with a cellular basis (the Murphy basis, see [17]) which has remarkable properties and one may ask, if the walled Brauer algebra has a cellular basis with similar properties:

- If a cell module of $R\mathfrak{S}_m$ is restricted to the subalgebra $R\mathfrak{S}_{m-1}$, then this module has a filtration by cell modules of $R\mathfrak{S}_{m-1}$. One can obtain this filtration canonically from the Murphy basis and the involved combinatorics are quite simple. Can one construct a basis of the walled Brauer algebra with the same property?
- The annihilator of $R\mathfrak{S}_m$ on the ordinary tensor space is a cell ideal ([10]). Is the same true for the annihilator of the walled Brauer algebra on the mixed tensor space? Or is there at least a cellular basis of the walled Brauer algebra such that for each n , a subset of this basis spans the annihilator? Or is the factor algebra at least again a cellular algebra?
- The ordinary tensor space has a filtration by cell modules of $R\mathfrak{S}_m$, in fact it has a filtration with U - $R\mathfrak{S}_m$ -bimodules which are tensor products of a certain U -module with a cell module for $R\mathfrak{S}_m$. Is the mixed tensor space filtered by corresponding cell modules/bimodules as well?

The known cellular bases do not have these properties and we will define a new cellular basis of the walled Brauer algebra, such that we can give positive answers to these three questions. Before we describe the answers, a subtle point has to be noted: there cannot exist a generic cellular structure on the walled Brauer algebra such that the mixed tensor space is filtered by cell modules with respect to this cellular structure. By ‘generic’ we mean a cellular structure for $R = \mathbb{Z}[x]$ which can be specialized to cellular structures for arbitrary R and x .

Consider for example the case $r = s = 2$. The walled Brauer algebra over $\mathbb{C}(x)$ with parameter x is semisimple, there are four irreducible modules of dimension 1, one of dimension 2 and one of dimension 4. Although, there might be different (generic) cellular structures on the walled Brauer algebra, the dimensions of the cell modules must coincide with the dimensions of the irreducible modules in the semisimple case. If $n = 2$ and $R = \mathbb{C}$, then the action of the walled Brauer algebra on the mixed tensor space is semisimple since the action of U is semisimple. In this case, the mixed

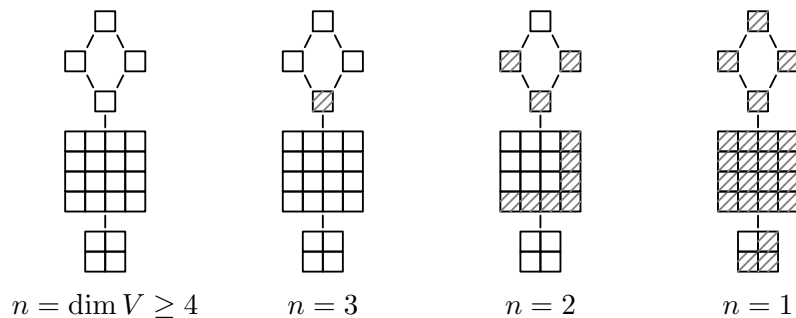


Figure 1

tensor space decomposes into a direct sum of three 3-dimensional, one 2-dimensional and five one-dimensional irreducible modules. Thus, it follows that the mixed tensor space does not have a cell filtration, since the only cell module that might have a 3-dimensional constituent is 4-dimensional, but this cell module is not semisimple. In the same way, the annihilator is not a cell ideal, otherwise the walled Brauer algebra modulo the annihilator would be again a cellular algebra and semisimple with irreducible modules of dimension 1, 2 or 4, a contradiction. A similar phenomenon has been observed in [11] for the action of the Brauer algebra on tensor space.

The example above shows that if a cellular basis of the walled Brauer algebra includes a basis of the annihilator (when x is specialized to n) then there might be elements λ in the poset (Λ, \leq) attached to the cell datum such that there are basis elements $C_{S,T}^\lambda$ which lie in the annihilator as well as basis elements which do not lie in the annihilator. Note that only for $x = n$ the walled Brauer algebra acts on some mixed tensor space and then the annihilator is defined. We will construct a (generic) basis of the walled Brauer algebra such that under specialization a basis element is in the annihilator iff S or T is not in a certain subset of $M(\lambda)$ (depending on n). Figure 1 illustrates this in the case $r = s = 2$. The big squares stand for the elements of the poset Λ . Each big square (which corresponds to $\lambda \in \Lambda$) consists of an array of small squares, the number of rows/columns is equal to the cardinality of $M(\lambda)$. So, each small square represents one basis element. The basis elements in the annihilator for various n are those in the shaded areas.

The example suggests that the walled Brauer algebra modulo its anni-

hilator is again a cellular algebra by taking cosets of the non-shaded basis elements, with cell datum obtained from those of the walled Brauer algebra by eventually omitting elements in the sets $M(\lambda)$. In fact, the constructed basis has this property and thus the factor algebra of the walled Brauer algebra modulo the annihilator is a cellular algebra. We show that we can filter the mixed tensor space by cell modules of the factor algebra.

Instead of this basis containing a basis of the annihilator, we use basis elements which can be easily defined and still have the property that a subset of the cosets of basis elements forms a cellular basis of the factor. Restriction of cell modules can be canonically described with this basis. Summarizing the main results,

- Theorem 22 establishes the existence of the cellular basis of the walled Brauer algebra,
- Theorem 26 shows that restriction of cell modules is canonical with respect to this basis.
- In Theorem 46, we give a filtration of the mixed tensor space by certain bimodules, and thus cell modules of the algebra of U -endomorphisms.

This paper is organized as follows: in Section 2, we supply details about the action of the walled Brauer algebra on the mixed tensor space and cellular algebras. In Section 3 we give a combinatorial description of the rank of the annihilator of the walled Brauer algebra on the mixed tensor space starting with the trivial U - (or $\mathbb{C}GL_n$ -) module and successively tensoring with V and V^* . This leads to appropriate cell data for the cellular structures of both the walled Brauer algebra and its image in the algebra of U -endomorphisms (for $x = n$). Similar to the symmetric group case, the elements of indexing sets in such a cell datum are paths of tuples of partitions. In Section 4 we replace these paths by triples of certain tableaux and define the new cellular basis of the walled Brauer algebra. Section 5 investigates the behaviour of cell modules under restriction to the subalgebra $B_{r,s-1}(x)$ for $s \geq 1$ or $B_{r-1,0}$ for $r \geq 1, s = 0$. It turns out that the isomorphism between a factor and a cell module of the subalgebra can be defined by mapping a basis to a basis. The corresponding map on the triples of tableaux labeling the basis elements can be described quite easily. In Section 6 we define another (weakly) cellular basis of the walled Brauer algebra such that for each n and x specialized to n , a subset of this basis is a basis of the annihilator. This shows that the

algebra of U -endomorphisms inherits the desired cellular structure. Section 7 recalls some facts about the ordinary tensor space which are needed to finally show in Section 8 that the mixed tensor space is filtered by cell modules and moreover by bimodules.

2. Preliminaries

Throughout, let r , s and m be fixed nonnegative integers. Furthermore, let n be a natural number. Let R be a commutative ring with 1 and $x \in R$.

2.1. The walled Brauer algebra

A *Brauer diagram* d is a graph with $2m$ vertices such that each vertex is connected to precisely one other vertex. Usually, the vertices are located in two rows, m vertices in an upper row and m vertices in a bottom row, and the edges are drawn inside of this rectangle (see Figure 2). Edges connecting vertices of one row are called *horizontal*, edges connecting vertices of different rows are called *vertical*.



Figure 2: A *Brauer diagram*

Definition 1 ([2]). *The Brauer algebra $B_m(x)$ is the R -algebra which is free as an R -module with basis $\{d \mid d \text{ a Brauer diagram with } 2m \text{ vertices}\}$. The multiplication is given by concatenation of diagrams, closed cycles are deleted by multiplying x (see Figure 3).*

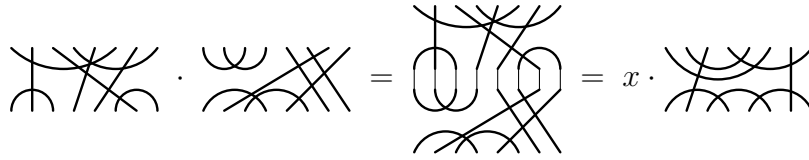


Figure 3: Multiplication in the Brauer algebra

Note that each permutation π of m letters corresponds to a Brauer diagram without horizontal edges, namely the diagram that connects the i -th

vertex in the upper row with the $\pi(i)$ -th vertex in the bottom row. Thus, the group algebra $R\mathfrak{S}_m$ is a subalgebra of the Brauer algebra.

Now let $m = r + s$. Consider a vertical wall between the r -th and $r + 1$ -st vertex in each row. A Brauer diagram is called *walled Brauer diagram* if all horizontal edges cross the wall and all vertical edges do not cross the wall (see Figure 4).

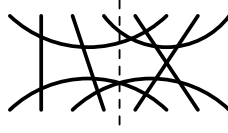


Figure 4: A walled Brauer diagram

Definition 2 ([1, 13, 20]). *The walled Brauer algebra $B_{r,s}(x)$ is the subalgebra of $B_m(x)$ generated by the walled Brauer diagrams. In fact, the walled Brauer diagrams form an R -basis of $B_{r,s}(x)$.*

Again, $R(\mathfrak{S}_r \times \mathfrak{S}_s)$ is a subalgebra of the walled Brauer algebra. Another way to decide whether or not a given Brauer diagram is a walled diagram (with r and s given) is the following: Draw arrows pointing downwards at the first r vertices in both rows. At the other vertices draw arrows pointing upwards. Then a Brauer diagram is a walled diagram if and only if the edges can be oriented consistently with the arrows at the vertices (see Figure 5).

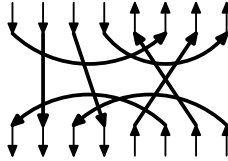


Figure 5: A walled Brauer diagram with orientation

Now, the notion of a walled Brauer diagram can be generalized to diagrams with not necessarily the same numbers of vertices in the top and bottom row. If two sequences \mathbf{I} and \mathbf{J} with entries in $\{\downarrow, \uparrow\}$ are given, then in a generalized Brauer diagram of type (\mathbf{I}, \mathbf{J}) the vertices in the top row are labeled by the entries of \mathbf{I} , the vertices in the bottom row are labeled by \mathbf{J} and the edges of the diagram again connect the vertices consistently with the arrows. Therefore, a walled Brauer diagram is the same as a generalized

diagram of type $((\downarrow^r, \uparrow^s), (\downarrow^r, \uparrow^s))$. If two diagrams of type (\mathbf{I}, \mathbf{J}) and (\mathbf{J}, \mathbf{K}) are given, then these diagrams can be concatenated to a diagram of type (\mathbf{I}, \mathbf{K}) .

2.2. The mixed tensor space

Let V be an R -free module of rank n with basis $\{v_1, \dots, v_n\}$. Let $I(n, m)$ be the set of m -tuples with entries in $\{1, \dots, n\}$. Then a basis of $V^{\otimes m}$ is given by $\{v_{\mathbf{i}} = v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_m} \mid \mathbf{i} = (i_1, \dots, i_m) \in I(n, m)\}$. The Brauer algebra $B_m(n)$ acts on $V^{\otimes m}$ which is called the *tensor space*. If d is a Brauer diagram, then the matrix (acting from the right) of the endomorphism induced by d with respect to the basis above is obtained in the following way: for $\mathbf{i}, \mathbf{j} \in I(n, m)$ write i_1, \dots, i_m at the vertices in the top row of the diagram and j_1, \dots, j_m at the vertices in the bottom row. The matrix entry at the position \mathbf{i}, \mathbf{j} is 1 if for all edges in d both ending vertices have the same number and 0 otherwise. This action extends the action of the symmetric group \mathfrak{S}_m permuting the components of the tensor product.

Let $V^* = \text{Hom}_R(V, R)$, which is as an R -module isomorphic to V . Let $\{v_i^* \mid i = 1, \dots, n\}$ be the basis of V^* dual to $\{v_i \mid i = 1, \dots, n\}$. By identifying v_i^* and v_i , the walled Brauer algebra $B_{r,s}(n)$ acts as a subalgebra of the Brauer algebra on $V^{\otimes r} \otimes V^{*\otimes s}$ called the *mixed tensor space*. More generally, a generalized walled Brauer diagram of type (\mathbf{I}, \mathbf{J}) induces a homomorphism from $V_{\mathbf{I}}$ to $V_{\mathbf{J}}$ where $V_{\mathbf{I}}$ is a tensor product of V 's and V^* 's such that \downarrow stands for V and \uparrow stands for V^* . Concatenation of those diagrams and deleting cycles by multiplication with n is compatible with composition of homomorphisms.

Let L be a linear combination of such generalized Brauer diagrams such that the number and orientation of vertices in the top row of each diagram coincide, the same for the bottom row. By a diagram involving a box containing L we mean the linear combination we get by taking the corresponding linear combination of diagrams obtained by replacing the box by the smaller diagram. We will use the term diagram also for diagrams containing boxes.

Example 3. Let $L = 3 \cdot \Psi - 2 \cdot \Upsilon$. Then

$$\boxed{L} = 3 \cdot \begin{array}{c} \text{diagram with crossing} \\ \downarrow \end{array} - 2 \cdot \begin{array}{c} \text{diagram with loop} \\ \downarrow \end{array}$$

Note that placing two diagrams next to each other corresponds to taking the tensor product of the corresponding maps. This has the following consequence:

Remark 4. *If a diagram contains a box inducing the zero map, then this diagram itself induces the zero map.*

If $R = \mathbb{C}$ the quantum group algebra $\mathbb{C}GL_n(\mathbb{C})$ also acts on the mixed tensor space and this action commutes with the action of the walled Brauer algebra. We have the following famous result:

Theorem 5 (Schur-Weyl duality, [1]). *Let $B_{r,s}(n)$ be the walled Brauer algebra over \mathbb{C} .*

1. *The \mathbb{C} -algebra homomorphism*

$$B_{r,s}(n) \rightarrow \text{End}_{\mathbb{C}GL_n(\mathbb{C})}(V^{\otimes r} \otimes V^{*\otimes s})$$

is surjective.

2. *The \mathbb{C} -algebra homomorphism*

$$\mathbb{C}GL_n(\mathbb{C}) \rightarrow \text{End}_{B_{r,s}(n)}(V^{\otimes r} \otimes V^{*\otimes s})$$

is surjective.

Let $U_{\mathbb{Z}}$ be the \mathbb{Z} -form of the universal enveloping algebra of the Lie algebra \mathfrak{gl}_n defined as in [3] and let $U = U_R = R \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$. Then it follows from the results in [4] and [5] for the special case $q = 1$, that this result holds more generally when \mathbb{C} is replaced by a commutative ring R with identity and the group algebra $\mathbb{C}GL_n(\mathbb{C})$ is replaced by U_R .

2.3. Cellular algebras

Definition 6. [8] *A cellular algebra over R is an R -algebra A together with a partially ordered set (Λ, \leq) , for each $\lambda \in \Lambda$ a finite set $M(\lambda)$ and an R -basis*

$$B = \{C_{S,T}^{\lambda} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$$

of A such that the following conditions hold:

- C1: The map $*$: $A \rightarrow A$: $C_{S,T}^{\lambda} \mapsto (C_{S,T}^{\lambda})^* = C_{T,S}^{\lambda}$ linearly extends to an anti-involution of A .*

C2: If $\lambda \in \Lambda$ and $T, T' \in M(\lambda)$ then for each $a \in A$ there exist $r_a(T, T') \in R$ such that

$$C_{S,T}^\lambda a \equiv \sum_{T' \in M(\lambda)} r_a(T, T') C_{S,T'}^\lambda \pmod{A(< \lambda)}$$

for $S \in M(\lambda)$ where $A(< \lambda)$ is the R -submodule of A generated by $\{C_{S'',T''}^\mu \mid \mu < \lambda, S'', T'' \in M(\mu)\}$

The basis B is called cellular basis.

Remark 7 ([7]). *Condition C1 in Definition 6 can be weakened to the following condition without losing the results of [8]:*

C1': There is an anti-involution $$: $A \rightarrow A$ such that*

$$C_{S,T}^{\lambda *} \equiv C_{T,S}^\lambda \pmod{A(< \lambda)}.$$

If A is an R -algebra satisfying the same conditions as a cellular algebra except for C1 which is replaced by C1', then we call A a weakly cellular algebra. If 2 is invertible in R , then a weakly cellular algebra is a cellular algebra.

One of the most important examples for cellular algebras is the group algebra of the symmetric group $R\mathfrak{S}_m$. Since we will use a cellular basis of this algebra, we recall the construction of such a basis due to Murphy ([17]).

A *composition* λ of m is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers whose sum is m . We write $\lambda \models m$. Repeated occurrences of the same integer are indicated by exponents. If λ and μ are compositions of m , we say that λ *dominates* μ and write $\lambda \triangleright \mu$ if for all $l \geq 1$ we have $\sum_{i=1}^l \lambda_i \geq \sum_{i=1}^l \mu_i$.

If λ is a composition of m such that $\lambda_1 \geq \lambda_2 \geq \dots$, then we say that λ is a *partition* of m and write $\lambda \vdash m$. The unique partition of 0 is denoted by \emptyset . The set of partitions of m is denoted by $\Lambda(m)$. The *Young diagram* $[\lambda]$ of a partition is the set $\{(i, j) \in \mathbb{N}^2 \mid 1 \leq j \leq \lambda_i, 1 \leq i\}$.

If λ is a partition of m , then a λ -*tableau* \mathbf{t} is a bijection $[\lambda] \rightarrow \{1, \dots, m\}$, λ is called the *shape* of \mathbf{t} and is denoted by $\text{shape}(\mathbf{t})$. Tableaux are often depicted as an array of boxes, one box for each element of $[\lambda]$ at the corresponding position (such that $(1, 1)$ is in the upper left corner) and we write the number corresponding to the position into the box. Figure 6 shows a tableau of shape $(4, 3, 2)$.

7	1	3	5
4	2	9	
8	6		

Figure 6: A $(4, 3, 2)$ -tableau

A tableau is called *row-standard* if the entries in each row are increasing from left to right, it is called *column-standard* if the entries in each column are increasing downwards and *standard* if it is both row- and column-standard. Let $\text{tab}(\lambda)$ be the set of λ -tableaux and $\text{Std}(\lambda)$ be the set of standard λ -tableaux. Let \mathbf{t}^λ be the λ -tableau where the numbers from 1 to m are written into the boxes row by row. Then \mathbf{t}^λ is a standard λ -tableau (see Figure 7).

1	2	3	4
5	6	7	
8	9		

Figure 7: \mathbf{t}^λ with $\lambda = (4, 3, 2)$

The symmetric group \mathfrak{S}_m acts from the right on the set of λ -tableaux by place permutation. If \mathbf{t} is a λ -tableau, let $d(\mathbf{t}) \in \mathfrak{S}_m$ be the unique element such that $\mathbf{t}.d(\mathbf{t}) = \mathbf{t}^\lambda$. Let \mathfrak{S}_λ be the row-stabilizer of \mathbf{t}^λ , so $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$. If $w \in \mathfrak{S}_m$, let $\text{sgn}(w)$ be the sign of the permutation w and $w^* = w^{-1}$. Then $*$ can be extended to an anti-automorphism of $R\mathfrak{S}_m$.

For a partition λ let $y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} \text{sgn}(w)w$. Furthermore, if β and \mathbf{t} are λ -tableaux, let $m_{\beta, \mathbf{t}}^\lambda = d(\beta)^* y_\lambda d(\mathbf{t})$. Then we have

Theorem 8 ([17]). *The group algebra $R\mathfrak{S}_m$ of the symmetric group is a cellular algebra with cellular basis*

$$\{m_{\beta, \mathbf{t}}^\lambda \mid \lambda \in \Lambda(m), \beta, \mathbf{t} \in \text{Std}(\lambda)\}.$$

In the notation of Definition 6, (Λ, \leq) is the set of partitions $(\Lambda(m), \geq)$ ordered by the dominance order. $M(\lambda)$ is the set $\text{Std}(\lambda)$ of standard λ -tableaux and the anti-automorphism $$ is defined above.*

Note that the elements $m_{\beta, \mathbf{t}}^\lambda$ are defined for arbitrary λ -tableaux which are not necessarily standard. The proof of Theorem 8 relies on the fact that the $m_{\beta, \mathbf{t}}^\lambda$ (with β, \mathbf{t} arbitrary λ -tableaux) clearly span $R\mathfrak{S}_m$. Thus one has

to show that if β or \mathbf{t} are not standard, then $m_{\beta, \mathbf{t}}^\lambda$ can be written as a linear combination of such elements involving tableaux which are ‘dominant’ with respect to some appropriate ordering.

It is easy to see and we will use this below, that if \mathbf{t} and \mathbf{t}' are λ -tableaux such that the set of entries of corresponding rows coincide, then $m_{\beta, \mathbf{t}}^\lambda = \pm m_{\beta, \mathbf{t}'}^\lambda$, the same holds for tableaux on the left side. In particular, each $m_{\beta, \mathbf{t}}^\lambda$ is a linear combination of $m_{\beta', \mathbf{t}'}^\lambda$ with β' and \mathbf{t}' row-standard.

Let \mathbf{t} be a row-standard λ -tableau for a composition λ of m . If $1 \leq i \leq m$, let $\mathbf{t} \downarrow i$ be the tableau obtained by restricting the corresponding bijection $\{1, \dots, m\} \leftrightarrow [\lambda]$ to $\{1, \dots, i\}$. So $\text{shape}(\mathbf{t} \downarrow i)$ is a composition of i . If β is a μ -tableau for some composition μ of m , we define $\mathbf{t} \supseteq \beta$ if and only if $\text{shape}(\mathbf{t} \downarrow i) \supseteq \text{shape}(\beta \downarrow i)$ for all $i = 1, \dots, m$. Then we have

Proposition 9 ([17]). *Let β and \mathbf{t} be row-standard λ -tableaux for a partition λ of m such that β is not standard. Then $m_{\beta, \mathbf{t}}^\lambda$ is congruent modulo $R\mathfrak{S}_n(\triangleright \lambda)$ to a linear combination of $m_{\beta', \mathbf{t}}^\lambda$ with $\beta' \triangleright \beta$. A similar statement holds if \mathbf{t} is not standard.*

Moreover, if β and \mathbf{t} are row-standard λ -tableaux, then $m_{\beta, \mathbf{t}}^\lambda$ is a linear combination of basis elements $m_{\beta', \mathbf{t}'}^\mu$ with β', \mathbf{t}' standard, $\beta' \supseteq \beta$ and $\mathbf{t}' \supseteq \mathbf{t}$.

We will construct a weakly cellular basis of $B_{r,s}(x)$ such that if x is specialized to n , then the annihilator $\text{ann}_{B_{r,s}(n)}(V^{\otimes r} \otimes V^{*\otimes s})$ of the walled Brauer algebra on the mixed tensor space has a basis consisting of a subset of this weakly cellular basis. In particular, $B_{r,s}(n)/\text{ann}_{B_{r,s}(n)}(V^{\otimes r} \otimes V^{*\otimes s})$ is again a weakly cellular algebra.

3. The rank of the annihilator

In this section, we will describe a combinatorial index set for a basis of the annihilator of the walled Brauer algebra on mixed tensor space. The next two propositions which hold also in the quantized case show that the annihilator is in fact R -free with R -free complement and the rank does not depend on R .

Proposition 10 ([10]). *The annihilator in the group algebra $R\mathfrak{S}_m$ of the symmetric group on $V^{\otimes m}$ is R -free with basis*

$$\{m_{\mathfrak{s}, \mathbf{t}}^\lambda \mid \lambda \in \Lambda(m), \lambda_1 > n, \mathfrak{s}, \mathbf{t} \in \text{Std}(\lambda)\}.$$

Remark 11. *In particular if $n' > n$, then $y_{(n')}$ acts as zero and hence each diagram with a box containing $y_{(n')}$ acts as zero.*

Proposition 12 ([4]). *There is an R -isomorphism between $R\mathfrak{S}_{r+s}$ and the walled Brauer algebra $B_{r,s}(n)$, that maps the annihilator in $R\mathfrak{S}_{r+s}$ on the tensor space bijectively to the annihilator in $B_{r,s}(n)$ on the mixed tensor space. In particular the annihilator in the walled Brauer algebra is as well R -free with an R -free complement.*

Thus, a basis of the annihilator of the walled Brauer algebra on mixed tensor space is indexed by pairs of standard tableaux for partitions λ with $\lambda_1 > n$. Anyhow, this combinatorial index set is related to the group algebra of the symmetric group and its action on tensor space, but not to the walled Brauer algebra. Propositions 10 and 12 show that it is enough to find a combinatorial description of the rank of the annihilator for $R = \mathbb{C}$.

The rank of the annihilator can be computed once we know the dimension of the image of the representation $B_{r,s}(n) \rightarrow \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes V^{*\otimes s})$ which is equal to $\text{End}_{\mathbb{C}GL_n(\mathbb{C})}(V^{\otimes r} \otimes V^{*\otimes s})$ by Theorem 5. To determine this dimension one can decompose the semisimple $\mathbb{C}GL_n(\mathbb{C})$ -module $V^{\otimes r} \otimes V^{*\otimes s}$ into simple submodules as in [19].

Definition 13. *Let $\lambda \vdash (r-k)$ and $\mu \vdash (s-k)$ for some nonnegative integer k . A path Y to (λ, μ) is a sequence $Y = (Y_0 = (\emptyset, \emptyset), Y_1, Y_2, \dots, Y_{r+s} = (\lambda, \mu))$ where $Y_i = (\lambda^{(i)}, \mu^{(i)})$ for $0 \leq i \leq r+s$ is a pair of partitions such that*

- *For $i \leq r$, $\mu^{(i)} = \emptyset$ is the empty partition and $[\lambda^{(i)}]$ is obtained from $[\lambda^{(i-1)}]$ by adding one box.*
- *For $r+1 \leq i \leq r+s$, either $\lambda^{(i)} = \lambda^{(i-1)}$ and $[\mu^{(i)}]$ is obtained from $[\mu^{(i-1)}]$ by adding one box or $\mu^{(i)} = \mu^{(i-1)}$ and $[\lambda^{(i)}]$ is obtained from $[\lambda^{(i-1)}]$ by removing one box.*

If Y is a path, we write $Y = (\emptyset \rightarrow Y_1 \rightarrow Y_2 \rightarrow \dots \rightarrow Y_{r+s})$. Given Y , let $\max(Y)$ be the maximum of the set $\{\lambda_1^{(i)} + \mu_1^{(i)} \mid i = 1, \dots, r+s\}$. We obviously have $\max(Y) \leq r+s$. To the tuples (λ, μ) of partitions with $\lambda_1 + \mu_1 \leq n$ one can define pairwise non-isomorphic simple rational $\mathbb{C}GL_n(\mathbb{C})$ -modules $V_{\lambda, \mu}$ such that

Proposition 14 ([19]). *We have:*

$$V^{\otimes r} \otimes V^{*\otimes s} \cong \bigoplus_{(\lambda, \mu)} V_{\lambda, \mu}^{\oplus n_{\lambda, \mu}}$$

where $n_{\lambda,\mu}$ is equal to the number of paths Y to (λ, μ) with $\max(Y) \leq n$.

Note that our partitions are the transposed partitions of Stembridge's partitions in [19]. Proposition 14 can be as well obtained using the Littlewood-Richardson rule ([14],[6]). Together with the previous results, we immediately obtain

Corollary 15. 1. *The algebra $\text{End}_U(V^{\otimes r} \otimes V^{*\otimes s})$ is R -free. Its rank is equal to the number of pairs (Y, Z) where Y, Z are paths to (λ, μ) for partitions $\lambda \vdash (r - k)$ and $\mu \vdash (s - k)$ with $\max(Y), \max(Z) \leq n$.*
2. *The rank of the walled Brauer algebra $B_{r,s}(x)$ is equal to the number of pairs (Y, Z) where Y, Z are paths to (λ, μ) with $\lambda \vdash (r - k)$ and $\mu \vdash (s - k)$.*
3. *The rank of the annihilator in the walled Brauer algebra on mixed tensor space is equal to the number of pairs (Y, Z) of paths to (λ, μ) such that $\max(Y) > n$ or $\max(Z) > n$.*

Proof. If M is a semisimple module over some \mathbb{C} -algebra A , and $M \cong \oplus_i S_i^{\otimes n_i}$ where the S_i are pairwise non-isomorphic simple modules, then we have $\dim_{\mathbb{C}} \text{End}_A(M) = \sum_i n_i^2$, thus the first part follows. Since the rank of the walled Brauer algebra does not depend on R and the parameter x and since the walled Brauer algebra acts faithfully for $n \geq r + s$ we obtain the second part. The rest follows. \square

4. A basis of the walled Brauer algebra

In Theorem 8, a basis of the symmetric group algebra was indexed by pairs of tableaux. These tableaux can be identified with paths of partitions. The other way around we will introduce tableaux replacing paths to (λ, μ) in this section.

Further, we define a basis of the walled Brauer algebra indexed by these tableaux. In order to obtain this we introduce a generalization of Young diagrams and tableaux.

We call subsets $[\rho]$ of \mathbb{N}^2 which can be written as $[\nu] \setminus [\lambda]$ for compositions ν, λ *skew diagrams*. Further we extend the term tableau to every injective map from a skew diagram $[\rho]$ into the integers and call such a tableau a ρ -tableau (here ρ is just a notation and has not the meaning of a sequence of certain numbers). As before these (skew-)tableaux get depicted by an array of boxes and the corresponding numbers in these boxes, see Figure 8. Given

a tableau \mathbf{t} we call the image of \mathbf{t} the filling of the tableau and denote it by $\text{fill}(\mathbf{t})$.

In an even wider context, ρ -tableaux can be defined as bijections from $[\rho]$ into a linearly ordered set. Then the terms row standard, standard and the dominance order on tableaux can be defined accordingly.

In particular, we consider maps to subsets of \mathbb{Z} ordered by \leq or \geq . If the subset of \mathbb{Z} is ordered by \leq , then row standard, etc. is defined as before. If the subset of \mathbb{Z} is ordered by \geq then row standard means that in each row the entries are decreasing from left to right. To avoid confusion we will call such tableaux anti-row standard and similarly a tableau is anti-standard if it is anti-row standard and the entries in each column are decreasing from top to bottom.

The dominance order \supseteq^{anti} on the set of anti-row standard tableaux then can be explained as follows: If β and \mathbf{t} are anti-row standard tableaux then $\mathbf{t} \supseteq^{\text{anti}} \beta$ if and only if $\text{shape}(\mathbf{t} \uparrow i) \supseteq^{\text{anti}} \text{shape}(\beta \uparrow i)$, where $\mathbf{t} \uparrow i$ denotes the restriction of the tableau to all integers greater or equal to i .

If \mathbf{t} is a tableau, let $d(\mathbf{t})$ be the permutation such that in $\mathbf{t}.d(\mathbf{t})$ the entries increase from left to right row by row and let $d^{\text{anti}}(\mathbf{t})$ be the permutation such that in $\mathbf{t}.d^{\text{anti}}(\mathbf{t})$ the entries decrease from left to right row by row.

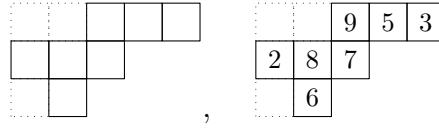


Figure 8: A *skew diagram* $[\rho]$ and a ρ -tableau

Recall, that r and s are two fixed non-negative integers. Let $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ be a triple of tableaux with the following properties:

- \mathbf{t} is a row-standard ν -tableau with entries $\{1, 2, \dots, r\}$ where ν is a partition of r ,
- \mathbf{u} is a row anti-standard ρ -tableau with k boxes, such that $[\rho] \subseteq [\nu]$; $[\nu] \setminus [\rho] = [\lambda]$ where λ is a composition of $r - k$,
- \mathbf{v} is a row-standard μ -tableau with μ a partition of $s - k$,
- the entries of \mathbf{u} and \mathbf{v} are $\{1, 2, \dots, s\}$.

We call such a triple $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ a row-standard triple, we call (λ, μ) its shape and $r + s$ its length. We call $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ standard if additionally:

- \mathbf{t} is standard,
- λ is a partition of $r - k$ and \mathbf{u} is anti-standard,
- \mathbf{v} is standard.

We denote the set of all pairs (λ, μ) of partitions of $r - k$ and $s - k$ resp. for some non-negative k by $\Lambda(r, s)$ and the set of all standard triples of shape (λ, μ) by $M(\lambda, \mu)$.

Proposition 16. *Let (λ, μ) be a pair of partitions in $\Lambda(r, s)$. There is a bijection between the set $M(\lambda, \mu)$ and the set of paths to (λ, μ) .*

Proof. To each path $Y = (\emptyset \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_{r+s})$ we relate an element of $M(\lambda, \mu)$ in the following way:

- Let $\nu = \lambda^{(r)}$, $\lambda = \lambda^{(r+s)}$, $\mu = \mu^{(r+s)}$ and ρ be defined by $[\rho] = [\nu] \setminus [\lambda]$.
- Let \mathbf{t} , \mathbf{u} and \mathbf{v} be the ν -, ρ - and μ -tableaux respectively satisfying:
 - For $1 \leq i \leq r$ if $[\lambda^{(i)}]$ is obtained from $[\lambda^{(i-1)}]$ by adding a box then i is the entry of this box inside \mathbf{t} .
 - For $1 \leq i \leq s$ if $[\mu^{(r+i)}]$ is obtained from $[\mu^{(r+i-1)}]$ by adding a box then i is the entry of this box inside \mathbf{v} .
 - For $1 \leq i \leq s$ if $[\lambda^{(r+i)}]$ is obtained from $[\lambda^{(r+i-1)}]$ by deleting a box then i is the entry of this box inside \mathbf{u} .

Obviously this is an 1-1-correspondence. □

Example 17. *Let Y be the path to $\left(\begin{smallmatrix} \square \\ \square \end{smallmatrix}, \square\square\square \right)$ with*

$$\begin{aligned}
Y = (\emptyset, \emptyset) &\rightarrow \left(\begin{smallmatrix} \square \\ \square \end{smallmatrix}, \emptyset \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}, \emptyset \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \emptyset \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square & \square & \square \end{smallmatrix}, \emptyset \right) \\
&\rightarrow \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \emptyset \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \square \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \square\square \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}, \square\square \right) \rightarrow \left(\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}, \square\square\square \right)
\end{aligned}$$

If we go through the procedure of the proof, then this path is related to the following standard triple $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ of tableaux

$$(\mathbf{t}, \mathbf{u}, \mathbf{v}) = \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline & 4 \\ \hline & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline \end{array} \right).$$

For each row-standard triple of tableaux $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ we will define a linear combination of generalized diagrams. Note that in general, these linear combinations are not elements of the walled Brauer algebra since they involve different numbers of vertices in the top and in the bottom row. Eventually, we will use these linear combinations to define a basis for the walled Brauer algebra.

We first define two additional tableaux \mathfrak{o} and β . The first tableau \mathfrak{o} is the ν -tableau with the entries of \mathbf{u} in the same positions as in \mathbf{u} . Into the remaining boxes we fill in different integers greater than s such that these integers decrease in order from left to right along the rows. The particular choice of the integers does not play a role. The second tableau β is row standard of shape $(k, s - k)$, such that the entries of the first row are the entries of \mathbf{u} and the entries of the second row are the entries of \mathbf{v} .

Let $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$ be the element defined via the diagram in figure 9. Note that there are k horizontal edges, $r - k$ down- and $s - k$ up-arrows in the top row and r down- and s up-arrows in the bottom row.

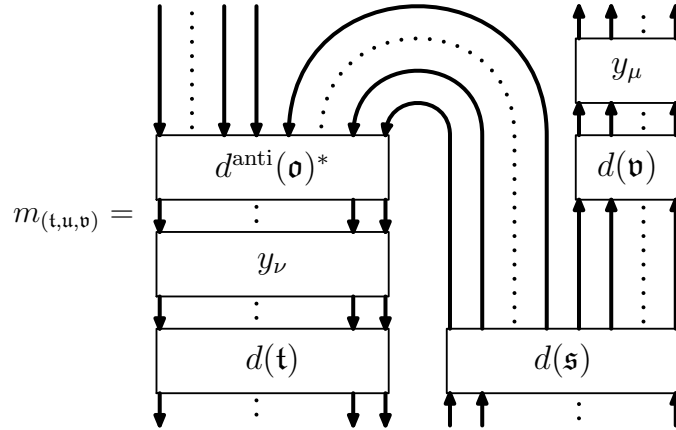


Figure 9: The element $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$

If the vertices in the bottom row of this diagram are labeled from left to right by $1, \dots, r$ and $1, \dots, s$ respectively and the $r - k$ vertices on the upper left are labeled by the additional entries in \mathfrak{o} , decreasing from left to right, then the element $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ can be easily illustrated: the entries of \mathfrak{t} , read row by row, indicate the ending vertices in the bottom left of the edges coming from y_ν . In the same way, the entries of \mathfrak{o} indicate the starting vertices of the edges ending in y_ν and the entries of \mathfrak{v} indicate the starting vertices of the edges ending in y_μ .

Example 18. Consider the standard triple

$$(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 9 & \\ \hline 7 & 8 & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & 9 & 1 \\ \hline & & 4 & \\ \hline & 6 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 5 & 7 \\ \hline 3 & & \\ \hline 8 & & \\ \hline \end{array} \right).$$

Then $\nu = (4, 3, 2)$, $\lambda = (2, 2, 1)$ and $\mu = (3, 1, 1)$. The additional tableaux are

$$\mathfrak{o} = \begin{array}{|c|c|c|c|} \hline 14 & 13 & 9 & 1 \\ \hline 12 & 11 & 4 & \\ \hline 10 & 6 & & \\ \hline \end{array} \quad \text{and} \quad \mathfrak{B} = \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 6 & 9 & \\ \hline 2 & 3 & 5 & 7 & 8 \\ \hline \end{array} \quad \text{and we get the diagram in Figure 10.}$$

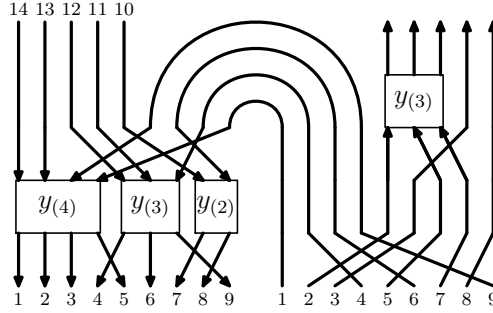


Figure 10: $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ as in Example 18

For $(\lambda, \mu) \in \Lambda(r, s)$ let $m_{\lambda, \mu}$ be the element

$$m_{\lambda, \mu} := \begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \boxed{y_\lambda} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \end{array} \quad \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \boxed{y_\mu} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \end{array}.$$

Let $\tau = (\lambda_1, 1^{(\nu_1 - \lambda_1)}, \lambda_2, 1^{(\nu_2 - \lambda_2)}, \dots)$ which is a composition of r , then $y_\nu = y_\tau \cdot z$ where z is an alternating sum of right coset representatives of \mathfrak{S}_τ in

\mathfrak{S}_ν . Note that $d^{\text{anti}}(\mathfrak{o})^* y_\tau = y_\lambda d^{\text{anti}}(\mathfrak{o})^*$. It follows that there is some diagram $b_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ such that $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} = m_{\lambda, \mu} \cdot b_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$.

We define

$$m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} := b_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')}^* m_{\lambda, \mu} b_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$$

which is an element of $B_{r, s}(x)$. Note that this definition does not depend on the choice of $b_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$.

Remark 19. *Theorem 8 and Proposition 9 still hold if $\text{Std}(\lambda)$ is replaced by the set of standard tableaux with entries in a fixed ordered set. In particular, $\text{Std}(\lambda)$ can be replaced by the set of anti-standard tableaux with a fixed filling. Then the basis elements have to be defined by $m_{\mathfrak{B}, \mathfrak{t}}^\lambda = d^{\text{anti}}(\mathfrak{B})^* y_\lambda d^{\text{anti}}(\mathfrak{t})$. One can even allow different sets of λ -tableaux for \mathfrak{B} and \mathfrak{t} .*

Suppose now that $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ are row-standard triples of shape (λ, μ) . Let \mathfrak{o} and \mathfrak{B} be the tableaux defined above for the triple $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and similarly \mathfrak{o}' and \mathfrak{B}' for the triple $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$. Let $m_{\mathfrak{o}, \mathfrak{t}}^\nu = d^{\text{anti}}(\mathfrak{o})^* y_\nu d(\mathfrak{t})$ and similarly $m_{\mathfrak{o}', \mathfrak{t}'}^\nu$ and $m_{\mathfrak{B}', \mathfrak{v}'}^\mu$ be defined in the above sense. Choose $b_{\mathfrak{o}, \mathfrak{t}}^\nu$ and $b_{\mathfrak{o}', \mathfrak{t}'}^\nu$ such that $m_{\mathfrak{o}, \mathfrak{t}}^\nu = y_\lambda b_{\mathfrak{o}, \mathfrak{t}}^\nu$ and $m_{\mathfrak{o}', \mathfrak{t}'}^\nu = y_\lambda b_{\mathfrak{o}', \mathfrak{t}'}^\nu$. Then $m_{\mathfrak{t}', \mathfrak{o}'}^\nu = (b_{\mathfrak{o}', \mathfrak{t}'}^\nu)^* y_\lambda$ and the element $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ is given in Figure 11.

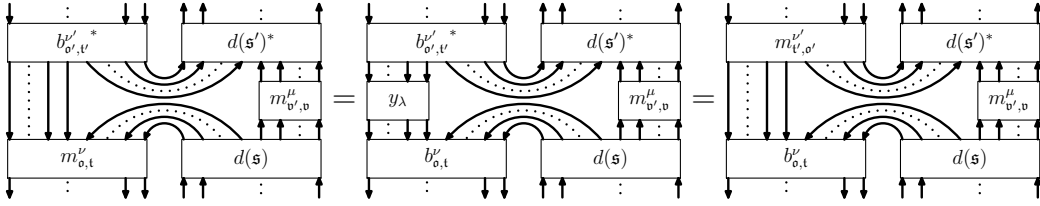


Figure 11: $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$

We will show that some of these elements form a cellular basis. The partially ordered set is $\Lambda(r, s)$ with the following ordering: Let $(\lambda, \mu), (\lambda', \mu') \in \Lambda(r, s)$. Then $\lambda \vdash (r - k)$, $\mu \vdash (s - k)$, $\lambda' \vdash (r - k')$ and $\mu' \vdash (s - k')$ for some k and k' . We define

$$(\lambda', \mu') \succeq (\lambda, \mu) :\Leftrightarrow k' > k \text{ or } (k' = k, \lambda' \succeq \lambda \text{ and } \mu' \succeq \mu).$$

For $(\lambda, \mu) \in \Lambda(r, s)$ let $\tilde{m}_{\lambda, \mu} \in B_{r, s}(x)$ be the element shown in Figure 12. This element coincides with $m_{\lambda, \mu}$ except for k additional horizontal edges on

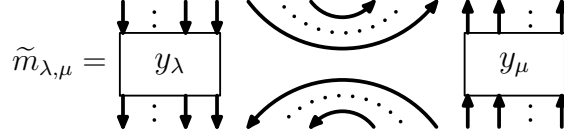


Figure 12: $\tilde{m}_{\lambda, \mu}$

the top and on the bottom.

Further, let $B^{\geq(\lambda, \mu)}$ be the ideal of $B_{r,s}(x)$ generated by all $\tilde{m}_{\lambda', \mu'}$ with $(\lambda', \mu') \geq (\lambda, \mu)$. We claim that as an R -module, $B^{\geq(\lambda, \mu)}$ is generated by all $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ where $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ and $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ are row-standard triples of shape (λ', μ') with $(\lambda', \mu') \geq (\lambda, \mu)$. By definition, these elements are elements of $B^{\geq(\lambda, \mu)}$. On the other hand, each element of $B^{\geq(\lambda, \mu)}$ is a linear combination of $b_1 \tilde{m}_{\lambda', \mu'} b_2$ with $(\lambda', \mu') \geq (\lambda, \mu)$ and b_1, b_2 walled Brauer diagrams. By deleting cycles, such an element can be written as (possibly a multiple of) $b'_1 m_{\lambda', \mu'} b'_2$ where b'_1 and b'_2 are generalized diagrams. If the diagram $b'_1 m_{\lambda', \mu'} b'_2$ has more horizontal edges than $\tilde{m}_{\lambda', \mu'}$, say k'' edges, then $b'_1 m_{\lambda', \mu'} b'_2$ can be written as a linear combination of $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ with $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ and $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ row standard of shape $((1^{r-k''}), (1^{s-k''})) \triangleright (\lambda', \mu')$. If the number of horizontal edges is the same in $\tilde{m}_{\lambda', \mu'}$ and $b_1 \tilde{m}_{\lambda', \mu'} b_2$, then $b'_1 m_{\lambda', \mu'} b'_2 = \pm m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ with $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ and $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ row standard of shape (λ', μ') .

We call the elements of $B^{\geq(\lambda, \mu)}$ (λ, μ) -elements. Note that $B^{\geq((1^{r-k}), (1^{s-k}))}$ is the ideal of the walled Brauer algebra generated by the walled Brauer diagrams with at least k horizontal edges on top and on bottom.

Suppose now that $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and $(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})$ are row-standard triples of shape (λ, μ) and $(\tilde{\lambda}, \tilde{\mu})$ respectively. Let \mathfrak{o} be defined as above and let $\tilde{\mathfrak{o}}$ be the corresponding tableau for $(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})$ such that \mathfrak{o} and $\tilde{\mathfrak{o}}$ have the same additional entries. Then we define $(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) \geq (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ if and only if

- $(\tilde{\lambda}, \tilde{\mu}) \triangleright (\lambda, \mu)$ or
- $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)$, $\tilde{\mathfrak{t}} \geq \mathfrak{t}$, $\tilde{\mathfrak{o}} \geq^{\text{anti}} \mathfrak{o}$ and $\tilde{\mathfrak{v}} \geq \mathfrak{v}$.

Lemma 20. *Let $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ and $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ be row-standard triples of shape (λ, μ) such that $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ is not standard. Then $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ is a linear combination of elements $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})}$ with $(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) \geq (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and of (λ', μ') -elements such that $(\lambda', \mu') \triangleright (\lambda, \mu)$.*

Proof. Recall that $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ can be depicted as in Figure 11. Suppose first that \mathfrak{v} is not standard. By Proposition 9, $m_{\mathfrak{v}', \mathfrak{v}}^{\mu}$ is a linear combination

of $m_{\mathbf{v}', \tilde{\mathbf{v}}}^{\tilde{\mu}}$ where either $\tilde{\mathbf{v}}' = \mathbf{v}'$ and $\tilde{\mathbf{v}} \triangleright \mathbf{v}$ (and thus $\tilde{\mu} = \mu$) or $\tilde{\mu} \triangleright \mu$. Replacing $m_{\mathbf{v}', \mathbf{v}}^{\mu}$ in $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ by this linear combination shows the result.

Now suppose that \mathbf{t} is not standard or \mathbf{o} is not anti-standard. Again by Proposition 9, $m_{\mathbf{o}, \mathbf{t}}^{\nu}$ is a linear combination of $m_{\tilde{\mathbf{o}}, \tilde{\mathbf{t}}}^{\tilde{\nu}}$ with $\tilde{\mathbf{o}}$ anti-standard and $\tilde{\mathbf{t}}$ standard, $\tilde{\mathbf{o}} \succeq^{\text{anti}} \mathbf{o}$ and $\tilde{\mathbf{t}} \succeq \mathbf{t}$ and $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ can be accordingly written as a linear combination. Note that λ is by definition the shape of $\mathbf{o} \uparrow s + 1$.

If $\text{shape}(\tilde{\mathbf{o}} \uparrow s + 1) \triangleright \text{shape}(\mathbf{o} \uparrow s + 1)$, then the element obtained by replacing $m_{\mathbf{o}, \mathbf{t}}^{\nu}$ by $m_{\tilde{\mathbf{o}}, \tilde{\mathbf{t}}}^{\tilde{\nu}}$ in $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ is a $(\tilde{\lambda}, \mu)$ -element where $\tilde{\lambda} = \text{shape}(\tilde{\mathbf{o}} \uparrow s + 1) \triangleright \lambda$.

If $\text{shape}(\tilde{\mathbf{o}} \uparrow s + 1) = \text{shape}(\mathbf{o} \uparrow s + 1) = \lambda$ then we have $\tilde{\mathbf{o}} \uparrow s + 1 = \mathbf{o} \uparrow s + 1$, since $\mathbf{o} \uparrow s + 1$ is the maximal row anti-standard λ -tableau with this filling. But then the element obtained by replacing $m_{\mathbf{o}, \mathbf{t}}^{\nu}$ by $m_{\tilde{\mathbf{o}}, \tilde{\mathbf{t}}}^{\tilde{\nu}}$ is $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ where $\tilde{\mathbf{u}}$ is obtained by restriction of $\tilde{\mathbf{o}}$. □

Remark 21. *The proof of Lemma 20 shows: if $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ appears in the linear combination in Lemma 20 and $\tilde{\mathbf{B}}$ is defined as above for $(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$, then $\tilde{\mathbf{B}} = \mathbf{B}$.*

Theorem 22. *The set*

$$\{m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M(\lambda, \mu), (\lambda, \mu) \in \Lambda(r, s)\}$$

is a cellular basis of the walled Brauer algebra $B_{r,s}(x)$. The partial order on $\Lambda(r, s)$ is given by \succeq .

Proof. Note that an analogue version of Lemma 20 for $(\mathbf{t}', \mathbf{u}', \mathbf{v}')$ not standard is also valid. Thus it follows from Lemma 20 that the $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ for standard triples $(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})$ of shape $(\lambda', \mu') \succeq (\lambda, \mu)$ generate $B^{\succeq(\lambda, \mu)}$. In particular, the set of all such elements generates $B_{r,s}(x) = B^{\succeq((1^r), (1^s))}$. Since the cardinality of the set of these elements is equal to the rank of $B_{r,s}(x)$, this set is linearly independent.

Recall the anti-involution $*$ of $B_{r,s}(x)$ given by reflecting diagrams at a horizontal axis. Then by definition, $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}^* = m_{(\mathbf{t}, \mathbf{u}, \mathbf{v}), (\mathbf{v}', \mathbf{u}', \mathbf{v}')}.$

It remains to prove the second property of a cellular algebra. Let $b \in B_{r,s}(x)$ and $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ and $(\mathbf{t}', \mathbf{u}', \mathbf{v}')$ be standard triples of shape (λ, μ) . Then $m_{(\mathbf{v}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})} b$ is an element of $B^{\succeq(\lambda, \mu)}$ and thus can be written as a linear combination of $m_{(\tilde{\mathbf{v}}', \tilde{\mathbf{u}}', \tilde{\mathbf{v}}'), (\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ for certain standard triples of shape dominating (λ, μ) by Lemma 20. If $m_{(\tilde{\mathbf{v}}', \tilde{\mathbf{u}}', \tilde{\mathbf{v}}'), (\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ appears in this linear combination, then

we have by construction that $(\tilde{\mathfrak{t}}', \tilde{\mathfrak{u}}', \tilde{\mathfrak{v}}') = (\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ or the shape of $(\tilde{\mathfrak{t}}', \tilde{\mathfrak{u}}', \tilde{\mathfrak{v}}')$ strictly dominates (λ, μ) .

Let $(\mathfrak{t}_0, \mathfrak{u}_0, \mathfrak{v}_0)$ be the unique standard triple of shape (λ, μ) such that $\nu_0 = (\lambda_1, \dots, \lambda_l, 1^k)$ is the shape of \mathfrak{t}_0 and $d(\mathfrak{t}_0)$, $d(\mathfrak{u}_0)$, $d(\mathfrak{v}_0)$ and $d^{\text{anti}}(\mathfrak{o}_0)$ are the identity.

For each standard triple $(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')$ there is a (linear combination of) walled Brauer diagram(s) d without horizontal edges such that $dm_{(\mathfrak{t}_0, \mathfrak{u}_0, \mathfrak{v}_0), (\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})} = m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})}$ for all standard triples $(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})$ and the result follows. \square

Remark 23. 1. The ideal $B^{\geq(\lambda, \mu)}$ is exactly the cell ideal $B_{r,s}(x)(\geq (\lambda, \mu))$.
 2. If $r = 0$ or $s = 0$, then $B_{r,s}(x)$ is the group algebra of a symmetric group and the basis of $B_{r,s}(x)$ we just defined coincides with the basis from Theorem 8.

5. Restriction

In this section we consider the restriction of cell modules to $B_{r,s-1}(x)$ or $B_{r-1,0}(x)$ for $s = 0$.

Definition 24. For every $(\lambda, \mu) \in \Lambda(r, s)$ let the cell module $C^{(\lambda, \mu)}$ be the right $B_{r,s}(x)$ -submodule of $B^{\geq(\lambda, \mu)} / B^{\triangleright(\lambda, \mu)}$ generated by $\tilde{m}_{\lambda, \mu} + B^{\triangleright(\lambda, \mu)}$ where $B^{\triangleright(\lambda, \mu)} = \bigcap_{(\lambda', \mu') \triangleright (\lambda, \mu)} B^{\geq(\lambda', \mu')}$. These modules in fact coincide with the cell modules in [8].

A basis of $C^{(\lambda, \mu)}$ is indexed by $M(\lambda, \mu)$ and in abuse of notation we use the set

$$\{m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \mid (\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in M(\lambda, \mu)\}$$

as a basis for $C^{(\lambda, \mu)}$.

Note that the action of the walled Brauer algebra on a cell module is again given by concatenation, deleting cycles by multiplication with x and factoring out diagrams involving $m_{\lambda', \mu'}$ with $(\lambda', \mu') \triangleright (\lambda, \mu)$.

If $s > 0$, then the walled Brauer diagrams of $B_{r,s-1}(x)$ can be embedded into $B_{r,s}(x)$ by adding a vertical edge connecting the rightmost vertex in the top row with the rightmost vertex in the bottom row. Similarly, $B_{r-1,0}(x)$ can be embedded into $B_{r,0}(x)$. We get a tower of algebras

$$R = B_{1,0}(x) \subset B_{2,0}(x) \subset \dots \subset B_{r,0}(x) \subset B_{r,1}(x) \subset \dots \subset B_{r,s}(x).$$

In this section, we describe the behavior of the cell modules under restriction in this tower of algebras. Fix r and s and let $\tilde{B} = B_{r-1,0}(x)$ if $s = 0$ and $\tilde{B} = B_{r,s-1}(x)$ if $s \geq 1$. For a $B_{r,s}(x)$ -module M we denote with $\text{Res}(M)$ its restriction to the subalgebra \tilde{B} .

Let $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ be a standard triple in $M(\lambda, \mu)$ for $(\lambda, \mu) \in \Lambda(r, s)$. First, suppose $s = 0$. Then \mathbf{u} and \mathbf{v} are empty tableaux. Let $\mathbf{t}' = \mathbf{t} \downarrow r - 1$, then $(\mathbf{t}', \emptyset, \emptyset)$ is a standard triple of shape $(\lambda', \mu) \in \Lambda(r - 1, 0)$. We call this triple the restriction of the triple $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ and denote it by $\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})$.

If $s \geq 1$ then we set $\mathbf{u}' := \mathbf{u} \downarrow (s - 1)$ and $\mathbf{v}' := \mathbf{v} \downarrow (s - 1)$. The triple $(\mathbf{t}, \mathbf{u}', \mathbf{v}')$ is in $M(\lambda', \mu')$ for $(\lambda', \mu') \in \Lambda(r, s - 1)$. More precisely $(\mathbf{t}, \mathbf{u}', \mathbf{v}')$ is either an element of $M(\lambda, \mu')$ and μ' is obtained from $[\mu]$ by removing a box, or $(\mathbf{t}, \mathbf{u}', \mathbf{v}')$ is an element of $M(\lambda', \mu)$ and $[\lambda']$ is obtained from $[\lambda]$ by adding a box. Again, we call $(\mathbf{t}, \mathbf{u}', \mathbf{v}')$ the restriction of the triple $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ and denote it by $\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})$.

For $(\lambda, \mu) \in \Lambda(r, s)$ we define the set $\text{Res}(\lambda, \mu)$ to be the set of tuples (λ', μ') occurring as shapes of restrictions of standard triples of shape (λ, μ) . So if $s = 0$, then $\text{Res}(\lambda, \emptyset) = \{(\lambda', \emptyset)\}$ where $[\lambda']$ is obtained from $[\lambda]$ by removing a box. If $s \geq 1$, then $\text{Res}(\lambda, \mu)$ is the set of tuples (λ', μ') obtained from (λ, μ) by either removing a box from $[\mu]$ or adding a box to $[\lambda]$ if $[\lambda]$ has less than r boxes.

We are now able to define for each $(\lambda', \mu') \in \text{Res}(\lambda, \mu)$ the following R -submodules of $C^{(\lambda, \mu)}$

$$U^{\supseteq(\lambda', \mu')} := \langle m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid \text{shape}(\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})) \supseteq (\lambda', \mu') \rangle_{R\text{-mod}},$$

$$U^{\triangleright(\lambda', \mu')} := \langle m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid \text{shape}(\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})) \triangleright (\lambda', \mu') \rangle_{R\text{-mod}}.$$

If $s = 0$ then μ is the empty partition and \mathbf{u} and \mathbf{v} are empty tableaux which can be omitted, e. g. $U^{\supseteq(\lambda', \emptyset)} = U^{\supseteq\lambda'}$. We have the classical result for the group algebra of the symmetric group $R\mathfrak{S}_r$:

Theorem 25 ([16]). *The modules $U^{\supseteq\lambda'}$ and $U^{\triangleright\lambda'}$ are $R\mathfrak{S}_{r-1}$ -submodules of $\text{Res } C^\lambda$. The modules $U^{\supseteq\lambda'}$ are linearly ordered by inclusion and give a filtration of the restricted cell module $\text{Res } C^\lambda$ with factors $U^{\supseteq\lambda'} / U^{\triangleright\lambda'} \cong C^{\lambda'}$. The isomorphism is given by $m_{\mathbf{t}} + U^{\triangleright\lambda'} \mapsto m_{\text{Res } \mathbf{t}}$.*

As the main result of this section we obtain a cell filtration of the restriction of cell modules.

Theorem 26. 1. *The modules $U^{\supseteq(\lambda', \mu')}$ and $U^{\triangleright(\lambda', \mu')}$ are \tilde{B} -submodules of $\text{Res } C^{(\lambda, \mu)}$.*

2. The set $\text{Res}(\lambda, \mu)$ is ordered linearly by dominance order. Thus the modules $U^{\triangleright(\lambda', \mu')}$ with $(\lambda', \mu') \in \text{Res}(\lambda, \mu)$ are accordingly ordered by inclusion and we get a filtration

$$\{0\} \subset U^{\triangleright(\lambda'_1, \mu'_1)} \subset U^{\triangleright(\lambda'_2, \mu'_2)} \subset \dots \subset U^{\triangleright(\lambda'_l, \mu'_l)} = \text{Res } C^{(\lambda, \mu)}$$

of $\text{Res } C^{(\lambda, \mu)}$ by \tilde{B} -modules.

3. Further a basis of each quotient $U^{\triangleright(\lambda', \mu')} / U^{\triangleright(\lambda', \mu')}$ is given by

$$\{m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} + U^{\triangleright(\lambda', \mu')} \mid \text{shape}(\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})) = (\lambda', \mu')\}.$$

4. The R -linear map $U^{\triangleright(\lambda', \mu')} / U^{\triangleright(\lambda', \mu')} \rightarrow C^{(\lambda', \mu')}$ given by

$$m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} + U^{\triangleright(\lambda', \mu')} \mapsto m_{\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})}$$

is an isomorphism of \tilde{B} -modules for every $(\lambda', \mu') \in \text{Res}(\lambda, \mu)$.

Proof. It can be seen easily, that $\text{Res}(\lambda, \mu)$ is ordered linearly. If $s = 0$ then the theorem follows from Theorem 25. So suppose $s \geq 1$. Let $(\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M(\lambda, \mu)$ and $b \in \tilde{B} = B_{r, s-1}(x)$. Suppose, the elements $\alpha_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} \in R$ are such that

$$m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \cdot b = \sum_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in M(\lambda, \mu)} \alpha_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} m_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}. \quad (1)$$

in $C^{(\lambda, \mu)}$. We have to show that $\alpha_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} \neq 0$ implies that $\text{shape}(\text{Res}(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})) \triangleright \text{shape}(\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})) = (\lambda', \mu')$ and that the following holds in $C^{(\lambda', \mu')}$:

$$m_{\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})} \cdot b = \sum_{\substack{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in M(\lambda, \mu) \\ \text{Res}(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = (\lambda', \mu')}} \alpha_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} m_{\text{Res}(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}. \quad (2)$$

Clearly, it is enough to show this claim for generators b of $B_{r, s-1}(x)$. It is well known that the walled Brauer algebra is generated by walled diagrams of the form $\downarrow \cdots \downarrow \times \downarrow \cdots \downarrow \uparrow \cdots \uparrow$, $\downarrow \cdots \downarrow \uparrow \cdots \uparrow \times \uparrow \cdots \uparrow$ and $\downarrow \cdots \downarrow \succ \uparrow \cdots \uparrow$. Recall the definition of the element $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$ in Figure 9.

If $b = \downarrow \cdots \downarrow \times \downarrow \cdots \downarrow \uparrow \cdots \uparrow \in B_{r, s-1}(x)$ is a basic transposition on the ‘left side’ of the wall, then

$$m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \cdot b = m_{(\mathbf{t}', \mathbf{u}, \mathbf{v})},$$

where \mathbf{t}' is obtained from \mathbf{t} by interchanging two entries. Let $\nu = \text{shape}(\mathbf{t})$. If $m_{\mathbf{t}'} = \sum_{\tilde{\mathbf{t}}} a_{\tilde{\mathbf{t}}} m_{\tilde{\mathbf{t}}}$ in C^ν then by the construction in the proof of Lemma 20 we have

$$m_{(\mathbf{t}', \mathbf{u}, \mathbf{v})} = \sum_{\tilde{\mathbf{t}}} a_{\tilde{\mathbf{t}}} m_{(\tilde{\mathbf{t}}, \mathbf{u}, \mathbf{v})}$$

in $C^{(\lambda, \mu)}$. So if $a_{(\tilde{\mathbf{t}}, \mathbf{u}, \mathbf{v})} = a_{\tilde{\mathbf{t}}} \neq 0$ then $\text{shape}(\tilde{\mathbf{t}}) = \text{shape}(\mathbf{t}) = \nu$ and thus $\text{shape}(\text{Res}(\tilde{\mathbf{t}}, \mathbf{u}, \mathbf{v})) = \text{shape}(\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v}))$. By the same considerations we have

$$m_{\text{Res}(\mathbf{t}', \mathbf{u}, \mathbf{v})} = \sum_{\tilde{\mathbf{t}}} a_{\tilde{\mathbf{t}}} m_{\text{Res}(\tilde{\mathbf{t}}, \mathbf{u}, \mathbf{v})}$$

in $C^{(\lambda', \mu')}$.

Suppose now that $b = \downarrow \cdot \downarrow \uparrow \cdot \uparrow \times \uparrow \cdot \uparrow \in B_{r, s-1}(x)$ is a basic transposition on the ‘right side’ of the wall, then

$$m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \cdot b = \pm m_{(\mathbf{t}, \mathbf{u}', \mathbf{v}')},$$

where $(\mathbf{u}', \mathbf{v}')$ is obtained from (\mathbf{u}, \mathbf{v}) by interchanging two entries and re-ordering the entries in each row. Keep in mind, that the position of s in the pair $(\mathbf{u}', \mathbf{v}')$ is the same as it is in (\mathbf{u}, \mathbf{v}) .

Suppose $a_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} \neq 0$. By construction $\tilde{\mathbf{v}} \supseteq \mathbf{v}'$ and $\tilde{\mathbf{o}} \supseteq^{\text{anti}} \mathbf{o}'$, where $\tilde{\mathbf{o}}$ and \mathbf{o}' are the additional tableaux to $(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ and $(\mathbf{t}', \mathbf{u}', \mathbf{v}')$ with the same additional entries.

In particular, we have $\text{shape}(\tilde{\mathbf{v}} \downarrow (s-1)) \supseteq \text{shape}(\mathbf{v}' \downarrow (s-1))$. Note that \mathbf{v} and \mathbf{v}' have the same shape and the position of s in \mathbf{v} and \mathbf{v}' is the same if s appears in \mathbf{v} . Thus we have $\text{shape}(\mathbf{v}' \downarrow (s-1)) = \text{shape}(\mathbf{v} \downarrow (s-1)) = \mu'$, so $\text{shape}(\tilde{\mathbf{v}} \downarrow (s-1)) \supseteq \mu'$.

Let \mathbf{o}_{Res} be the additional tableau for the triple $\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})$. Similarly, let $\tilde{\mathbf{o}}_{\text{Res}}$ and \mathbf{o}'_{Res} be defined. Since $\tilde{\mathbf{o}} \supseteq^{\text{anti}} \mathbf{o}'$ we have $\tilde{\mathbf{o}}_{\text{Res}} \supseteq^{\text{anti}} \mathbf{o}'_{\text{Res}}$ and thus $\text{shape}(\tilde{\mathbf{o}}_{\text{Res}} \uparrow s) \supseteq^{\text{anti}} \text{shape}(\mathbf{o}'_{\text{Res}} \uparrow s) = \text{shape}(\mathbf{o}_{\text{Res}} \uparrow s) = \lambda'$. It follows that $\text{shape}(\text{Res}(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})) = (\text{shape}(\tilde{\mathbf{o}}_{\text{Res}} \uparrow s), \text{shape}(\tilde{\mathbf{v}} \downarrow (s-1))) \supseteq \text{shape}(\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})) = (\lambda', \mu')$. Equation (2) follows by the classical result and the proof of Lemma 20.

Now let $b = e = \downarrow \cdot \downarrow \asymp \uparrow \cdot \uparrow \in B_{r, s-1}(x)$. Note that in this case, $s \geq 2$. At first sight the action of e seems to be more involved than the action of the transpositions, but it can also be described combinatorially if we do a case-by-case analysis. We have to take a look at the positions of r in \mathbf{t} and 1 in \mathbf{u} or \mathbf{v} and distinguish three cases.

- First case: r is an entry in row l of \mathbf{t} , 1 is an entry in row l of \mathbf{u} .

Denote the other entries in row l of \mathbf{t} with a_1, \dots, a_k . So we have

$$(\mathbf{t}, \mathbf{u}, \mathbf{v}) = \left(\begin{array}{c} \boxed{a_1 \mid a_2 \mid \dots \mid a_k \mid r} \\ \text{---} \end{array}, \begin{array}{c} \boxed{1} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \end{array} \right).$$

This means that in $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$, the box containing $y_{(k+1)}$ in y_ν is connected to the vertices in the bottom row indexed by r on the left and 1 on the right. It can be verified by direct computation that $\begin{array}{c} \downarrow \\ \vdots \\ \downarrow \end{array} \boxed{y_{(k+1)}} \begin{array}{c} \uparrow \\ \vdots \\ \uparrow \end{array} = (x-k) \cdot \begin{array}{c} \downarrow \\ \vdots \\ \downarrow \end{array} \boxed{y_{(k)}} \begin{array}{c} \uparrow \\ \vdots \\ \uparrow \end{array}$.

Thus we get $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \cdot e = (x-k) \cdot m_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ with

$$(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \left(\begin{array}{c} \boxed{a_1 \mid a_2 \mid \dots \mid a_k} \\ \text{---} \\ \boxed{r} \end{array}, \begin{array}{c} \boxed{1} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \end{array} \right)$$

which is a standard triple. In the same way we get the equation $m_{\text{Res}(\mathbf{t}, \mathbf{u}, \mathbf{v})} \cdot e = (x-k) \cdot m_{\text{Res}(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ in $C^{(\lambda', \mu')}$.

- Second case: 1 is an entry in row l of \mathbf{u} and r is not in row l of \mathbf{t} .

Denote the entries in row l of \mathbf{t} with a_1, \dots, a_k , then we have

$$(\mathbf{t}, \mathbf{u}, \mathbf{v}) = \left(\begin{array}{c} \boxed{a_1 \mid a_2 \mid \dots \mid a_k} \\ \text{---} \\ \boxed{r} \end{array}, \begin{array}{c} \boxed{1} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \end{array} \right).$$

This means that in $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$ the vertices in the bottom row labeled by r and by 1 are connected to different boxes inside y_ν . Note that $y_{(k)} = \sum_d \pm y_{(k-1)} d$ where d runs through a set of right coset representatives

of \mathfrak{S}_{k-1} in \mathfrak{S}_k . For every $1 \leq i \leq k$ define

$$(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}') = \left(\begin{array}{c} \boxed{\dots \hat{a}_i \dots} \\ \boxed{a_i} \\ \boxed{r} \end{array}, \begin{array}{c} \boxed{1} \end{array} \right)$$

where \hat{a}_i means omitting a_i . Then $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \cdot e = \sum_i \pm m_{(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}')}$ and similarly in $C^{(\lambda', \mu')}$ we have $m_{\text{Res}(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \cdot e = \sum_i \pm m_{\text{Res}(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}')}$. It might happen that $(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}')$ is not standard. Using the same arguments as we did for the generators without horizontal edges, $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \cdot e$ and $m_{\text{Res}(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \cdot e$ can be respectively written as a linear combination of basis elements and the claim follows by the same arguments as before.

- Third case: r is an entry in row l of \mathfrak{t} and 1 is an entry of \mathfrak{v} .

Let the entries in row l of \mathfrak{u} be b_1, \dots, b_k . We have

$$(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = \left(\begin{array}{c} \boxed{r} \\ \boxed{b_1 \ b_2 \ \dots \ b_k} \\ \boxed{1} \end{array} \right).$$

For every $1 \leq i \leq k$ define

$$(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}') = \left(\begin{array}{c} \boxed{\dots \hat{b}_i \dots} \\ \boxed{b_i} \\ \boxed{r} \end{array}, \begin{array}{c} \boxed{1} \end{array} \right).$$

Note that $\text{shape}(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}') = (\lambda, \mu)$. We have $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \cdot e = \sum_i \pm m_{(\mathfrak{t}'_i, \mathfrak{u}', \mathfrak{v}')}$. Note the special subcase if row l in \mathfrak{u} is empty. In this case e acts as zero on $m_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$. Again the claim follows. □

6. A basis adapted to the annihilator

In this section, we define another basis of the walled Brauer algebra. Again, this basis is indexed by tuples of standard triples. However, if x is specialized to n , then a subset of this basis is a basis for the annihilator on the mixed tensor space.

From Corollary 15 we see that $\max(Y)$ for a path Y plays an important role in a combinatorial description for a basis of the annihilator.

Definition 27. Let $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ be a standard triple of shape (λ, μ) . For $i = 0, \dots, s$ let u_i be the number of entries in the first row of \mathbf{u} which are $> i$ and let v_i be the number of entries in the first row of \mathbf{v} which are $\leq i$. Let $m_i = \lambda_1 + u_i + v_i$. Then let $\max(\mathbf{t}, \mathbf{u}, \mathbf{v})$ be the maximum of $\{m_0, \dots, m_s\}$.

If Y is a path and $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ is the corresponding standard triple, then $\max(Y) = \max(\mathbf{t}, \mathbf{u}, \mathbf{v})$.

Lemma 28. 1. Let $R = \mathbb{Z}[x]$. Let $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ be a standard triple of shape (λ, μ) with $\lambda \vdash (r - k)$ and $\mu \vdash (s - k)$. Then for each standard triple $(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ of shape (λ, μ) there exists a coefficient $r_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} \in R$ such that

- $r_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} \neq 0$ only if $\mathfrak{B} \triangleright^{\text{anti}} \tilde{\mathfrak{B}}$ (defined as above) or $(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = (\mathbf{t}, \mathbf{u}, \mathbf{v})$,
- $r_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} = 1$,
- if $(\mathbf{t}', \mathbf{u}', \mathbf{v}')$ is a standard triple of shape (λ, μ) then there exists an element $b \in B^{\triangleright(\lambda, \mu)}$ such that

$$\sum r_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} m_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})} + b \quad (3)$$

can be written as a linear combination of elements each of which involves $y_{(\max(\mathbf{t}, \mathbf{u}, \mathbf{v}))}$.

Note that the coefficients depend on $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ but not on $(\mathbf{t}', \mathbf{u}', \mathbf{v}')$. We write $r_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}^{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$ instead of $r_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ to emphasize that the coefficients depend on $(\mathbf{t}, \mathbf{u}, \mathbf{v})$.

2. If R is arbitrary, then the element in (3) is defined by specializing coefficients. In particular, if $x = n = \text{rank}(V) < \max(\mathbf{t}, \mathbf{u}, \mathbf{v})$ then this element is an element of the annihilator of the walled Brauer algebra on mixed tensor space.

Proof. The second part follows by Remark 11, so let $R = \mathbb{Z}[x]$. Since the upper part of the diagrams is not involved in the calculations, we show a similar result for $m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$ instead of $m_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$, then the lemma follows by premultiplying $b_{(\mathbf{t}', \mathbf{u}', \mathbf{v}')}^*$. We write \max instead of $\max(\mathbf{t}, \mathbf{u}, \mathbf{v})$ and choose i_0 such that $m_{i_0} = \max$.

Let $u = u_{i_0}$ and $v = v_{i_0}$. Then we have $\max = \lambda_1 + u + v$. Note that $\lambda_1 + u \leq \nu_1$ and $v \leq \mu_1$. By choosing coset representatives we get $z_1 \in R\mathfrak{S}_r$ and $z_2 \in R\mathfrak{S}_{s-k}$ such that

Thus we have

There are $\lambda_1 + u$ edges ending in $y_{(\lambda_1+u)}$, λ_1 of these edges start at the first λ_1 upper vertices on the left, the other edges start at the vertices on the right in the bottom row namely at the vertices numbered by the first u entries in the first row of \mathbf{u} .

Note that

where the sum runs over a set of right coset representatives of $\mathfrak{S}_{\lambda_1+u} \times \mathfrak{S}_v$ in \mathfrak{S}_{\max} containing the identity id . Let m_d be the element obtained from

$m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$ by adding a box filled with the permutation d below the boxes $y_{(\lambda_1 + u)}$ and $y_{(v)}$ in the right diagram. Then $\sum_d \text{sgn}(d) m_d$ contains $y_{(\max)}$. Certainly if $d = \text{id}$, then $m_d = m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}$.

Suppose first, that d connects the first λ_1 edges coming from $d^{\text{anti}}(\mathbf{o})^*$ to the box $y_{\lambda_1 + u}$. Then m_d is up to a sign equal to $m_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}$ where $\tilde{\mathbf{t}} = \mathbf{t}$ and $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are obtained from \mathbf{u} and \mathbf{v} by a permutation on the set of the first u entries in the first row of \mathbf{u} together with the first v entries in the first row of \mathbf{v} . Since these entries in \mathbf{u} are $> i_0$ and those in \mathbf{v} are $\leq i_0$ we have $\beta \triangleright^{\text{anti}} \tilde{\beta}$ for $d \neq \text{id}$. If $(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ is not yet standard then one can rewrite it as a linear combination of standard elements still satisfying $\beta \triangleright^{\text{anti}} \tilde{\beta}$ (c. f. Remark 21) and strictly dominating terms.

Suppose now that d connects one of the first λ_1 edges coming from $d^{\text{anti}}(\mathbf{o})^*$ to the box $y_{(v)}$. Then this edge is an additional horizontal edge and we have $b_{(\mathbf{t}', \mathbf{u}', \mathbf{v}')}^* m_d \in B^{\triangleright(\lambda, \mu)}$. □

Definition 29. Lemma 28 shows that for each pair $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ and $(\mathbf{t}', \mathbf{u}', \mathbf{v}')$ of standard triples there is an element $b \in B^{\triangleright(\lambda, \mu)}$, such that

$$\sum_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}), (\tilde{\tilde{\mathbf{t}}}, \tilde{\tilde{\mathbf{u}}}, \tilde{\tilde{\mathbf{v}}})} r_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})}^{(\mathbf{t}', \mathbf{u}', \mathbf{v}')} r_{(\tilde{\tilde{\mathbf{t}}}, \tilde{\tilde{\mathbf{u}}}, \tilde{\tilde{\mathbf{v}}})}^{(\mathbf{t}, \mathbf{u}, \mathbf{v})} m_{(\tilde{\mathbf{t}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}), (\tilde{\tilde{\mathbf{t}}}, \tilde{\tilde{\mathbf{u}}}, \tilde{\tilde{\mathbf{v}}})} + b \quad (4)$$

involves $y_{(\max(\mathbf{t}, \mathbf{u}, \mathbf{v}))}$. Similarly, b can be chosen such that the element in (4) involves $y_{(\max(\mathbf{t}', \mathbf{u}', \mathbf{v}'))}$.

For each pair of standard triples let the element $c_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ be one of these two elements such that $c_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ involves $y_{(\max)}$ where \max is the maximum of $\max(\mathbf{t}, \mathbf{u}, \mathbf{v})$ and $\max(\mathbf{t}', \mathbf{u}', \mathbf{v}')$. Note that this definition then makes sense for all commutative rings R with one.

Theorem 30. 1. Let R be a commutative ring with one. Then

$$\{c_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M(\lambda, \mu), (\lambda, \mu) \in \Lambda(r, s)\}$$

is a weakly cellular basis of the R -algebra $B_{r,s}(x)$ with anti-involution $*$.

2. If R is such that $x = n$, then the annihilator of $B_{r,s}(n)$ on mixed tensor space is free with basis

$$\left\{ c_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid \begin{array}{l} (\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M(\lambda, \mu), (\lambda, \mu) \in \Lambda(r, s), \\ \max(\mathbf{t}, \mathbf{u}, \mathbf{v}) \text{ or } \max(\mathbf{t}', \mathbf{u}', \mathbf{v}') > n \end{array} \right\}.$$

3. For $x = n$ let $\text{ann} = \text{ann}_{B_{r,s}(n)}(V^{\otimes r} \otimes V^{*\otimes s})$ be the annihilator of the walled Brauer algebra on mixed tensor space and let $\bar{c}_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \in B_{r,s}(n)/\text{ann}$ be the coset of $c_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ modulo the annihilator. Then

$$\left\{ \bar{c}_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \mid \begin{array}{l} (\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in M(\lambda, \mu), (\lambda, \mu) \in \Lambda(r, s), \\ \max(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}), \max(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}') \leq n \end{array} \right\}$$

is a weakly cellular basis of the R -algebra $B_{r,s}(x)/\text{ann} \cong \text{End}_U(V^{\otimes r} \otimes V^{*\otimes s})$.

Proof. The base change matrix between the two bases $\{c_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}\}$ and $\{m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}\}$ is uni-triangular since $r_{(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}})}^{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \neq 0$ implies that $\beta \triangleright^{\text{anti}} \tilde{\beta}$ or $(\tilde{\mathfrak{t}}, \tilde{\mathfrak{u}}, \tilde{\mathfrak{v}}) = (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and $r_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}^{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} = 1$. Thus $\{c_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}\}$ is in fact a basis. Since $c_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}^* - c_{(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}), (\mathfrak{t}', \mathfrak{u}', \mathfrak{v}')} \in B^{\triangleright(\lambda, \mu)}$ by Equation (4), Condition C1' is satisfied. Condition C2 can be easily verified.

The second part then follows since the basis elements are in fact elements of the annihilator, the basis has the right cardinality and the annihilator is R -free with R -free complement. The third part is a direct consequence. \square

Example 31. Recall the example in Figure from the introduction for $r = s = 2$. The elements of $\Lambda(2, 2)$ are $((2), (2))$, $((1^2), (2))$, $((2), (1^2))$, $((1^2), (1^2))$ (for $k = 0$), $((1), (1))$ (for $k = 1$) and (\emptyset, \emptyset) for $(k = 2)$, from top to bottom. The reader may list all triples of this shape and the corresponding maximum to obtain the diagrams in the figure.

As a direct consequence of Theorem 30 we get

Theorem 32. Let $\bar{m}_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ be the coset of $m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})}$ modulo ann . Then

$$\left\{ \bar{m}_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} \mid \begin{array}{l} (\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in M(\lambda, \mu), (\lambda, \mu) \in \Lambda(r, s), \\ \max(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}), \max(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}') \leq n \end{array} \right\}$$

is a cellular basis of $B_{r,s}(x)/\text{ann} \cong \text{End}_U(V^{\otimes r} \otimes V^{*\otimes s})$ with anti-involution induced by $*$.

Proof. Let

$$\tilde{c}_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} = \begin{cases} m_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} & \text{if } \max(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}), \max(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}') \leq n \\ c_{(\mathfrak{t}', \mathfrak{u}', \mathfrak{v}'), (\mathfrak{t}, \mathfrak{u}, \mathfrak{v})} & \text{otherwise.} \end{cases}$$

Again, $\{\tilde{c}_{(\nu, u', \mathfrak{v}'), (\mathfrak{t}, u, \mathfrak{v})}\}$ is a basis since the base change matrix is uni-triangular. A subset of this basis is a basis of the annihilator and thus this set is actually a basis of $B_{r,s}(x)/\text{ann}$. Condition C1 is obvious, Condition C2 follows using Equation (3) inductively. \square

Remark 33. Note that the partially ordered set corresponding to these cellular bases of $B_{r,s}(x)/\text{ann}$ is $\Lambda(r, s)$. The set corresponding to an element (λ, μ) of $\Lambda(r, s)$ is a subset of $M(\lambda, \mu)$, namely the set of standard triples $(\mathfrak{t}, u, \mathfrak{v})$ with $\max(\mathfrak{t}, u, \mathfrak{v}) \leq n$. We denote this set by $M_0(\lambda, \mu)$. Note that this set might be empty. To be precise, $M_0(\lambda, \mu) = \emptyset$ if and only if $\lambda_1 + \mu_1 > n$.

Let $\Lambda_0(r, s)$ be the set of pairs $(\lambda, \mu) \in \Lambda(r, s)$ such that $\lambda_1 + \mu_1 \leq n$. Then the cellular basis of $\text{End}_U(V^{\otimes r} \otimes V^{*\otimes s})$ is indexed by pairs of elements of $M_0(\lambda, \mu)$ with $(\lambda, \mu) \in \Lambda_0(r, s)$ and we can take $\Lambda_0(r, s)$ as corresponding partially ordered set.

7. A basis and a filtration for ordinary tensor space

Before we turn back to the walled Brauer algebra and mixed tensor space, we state some results on ordinary tensor space adjusted for our purposes. Let m be a natural number, then $V^{\otimes m}$ is called the *ordinary tensor space*.

Definition 34. Let λ be a partition of m and μ a composition of m into n parts. A λ -tableau of type μ is a filling of the boxes of $[\lambda]$ with not necessarily distinct numbers such that the number of entries equal to i is μ_i .

A λ -tableau of type μ is called semi-standard if it is row-standard and the entries weakly increase along the columns (see Figure 13). Let \mathcal{T} be the set of all tableaux of some type, let $\mathcal{T}(\lambda)$ be the set of λ -tableaux of some type, and let $\mathcal{T}_0(\lambda)$ denote the set of semi-standard λ -tableaux of some type.

1	2	3
1	2	4
2	4	
4	5	

Figure 13: A semi-standard $(3, 3, 2, 2)$ -tableau of type $(2, 3, 1, 3, 1)$

Definition 35. Let $\lambda \vdash m$ be a partition of m , $T \in \mathcal{T}(\lambda)$ and $\mathbf{t} \in \text{tab}(\lambda)$ be row-standard tableaux. Let $v_T = v_{\mathbf{i}} \in V^{\otimes m}$ where \mathbf{i} is the multi-index obtained from T by reading the entries row by row and let $v_{T\mathbf{t}} := v_{Ty_\lambda d(\mathbf{t})} \in V^{\otimes m}$.

Note that $v_{T\mathbf{t}}$ is a linear combination of those $v_{\mathbf{i}}$ with coefficients 1 or -1 where \mathbf{i} is a multi-index such that for each row in $[\lambda]$ the numbers in \mathbf{t} in this row indicate the positions of the numbers in T in this row.

Definition 36. If $T, S \in \mathcal{T}$ are row-standard, we say that T dominates S ($T \trianglerighteq S$) if the shape of $T \downarrow i$ dominates the shape of $S \downarrow i$ for all $i = 1, \dots, n$. Here $T \downarrow i$ is the tableau obtained from T by removing all boxes with entries greater than i .

The following result can be shown by similar methods as the results in [17] where a basis for the Hecke algebra of the symmetric group with similar properties is given.

Theorem 37. 1. The set

$$\{v_{T\mathbf{t}} \mid \lambda \vdash m, T \in \mathcal{T}_0(\lambda), \mathbf{t} \in \text{Std}(\lambda)\}$$

is a basis of $V^{\otimes m}$.

2. Let $\lambda \vdash m$, $T \in \mathcal{T}(\lambda)$ and $\mathbf{t} \in \text{tab}(\lambda)$ such that T and \mathbf{t} are row-standard but T is not semi-standard. Then $v_{T\mathbf{t}}$ is a linear combination of $v_{S\mathbf{B}}$ where either $\mathbf{B} = \mathbf{t}$ and $S \triangleright T$ or S and \mathbf{B} are tableaux of shape λ' where $\lambda' \triangleright \lambda$.
3. Let $\lambda \vdash m$, $T \in \mathcal{T}(\lambda)$ and $\mathbf{t} \in \text{tab}(\lambda)$ with T and \mathbf{t} row-standard, but now suppose that \mathbf{t} is not standard. Then $v_{T\mathbf{t}}$ is a linear combination of $v_{S\mathbf{B}}$ where either $\mathbf{B} \triangleright \mathbf{t}$ and $S = T$ or S and \mathbf{B} are tableaux of shape λ' where $\lambda' \triangleright \lambda$.

Proof. That the set in 1. is a generating system can be seen similarly as the results in [17]: obviously, the set of all $v_{T\mathbf{t}}$ with T and \mathbf{t} row-standard λ -tableaux for partitions λ generate the tensor space (all vectors $v_{\mathbf{i}}$ can be written as $v_{T\mathbf{t}}$ with $\lambda = (1^m)$).

If T is not semi-standard or \mathbf{t} is not standard then using Garnir relations $v_{T\mathbf{t}}$ can be rewritten as a linear combination similar as in 2. or 3. except that \triangleright should be replaced by $>$ where $>$ is the lexicographical order.

To show that this set is linearly independent and that the other two parts of the theorem hold, it suffices to find a bilinear form $\langle \cdot, \cdot \rangle : V^{\otimes m} \times V^{\otimes m} \rightarrow R$

and vectors $w_{T,\mathbf{t}} \in V^{\otimes m}$ for $T \in \mathcal{T}_0(\lambda)$, $\mathbf{t} \in \text{Std}(\lambda)$, $\lambda \vdash m$, such that the following holds: If $\lambda, \lambda' \vdash m$, $T \in \mathcal{T}_0(\lambda)$, $\mathbf{t} \in \text{Std}(\lambda)$, $S \in \mathcal{T}(\lambda')$ row-standard and $\beta \in \text{tab}(\lambda')$ row-standard, then we have

- $\langle w_{T,\mathbf{t}}, v_{T,\mathbf{t}} \rangle$ is invertible.
- $\langle w_{T,\mathbf{t}}, v_{S,\beta} \rangle \neq 0 \Rightarrow T \supseteq S$ and $\mathbf{t} \supseteq \beta$.

Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $V^{\otimes m}$ given by $\langle v_{\mathbf{i}}, v_{\mathbf{j}} \rangle = \delta_{\mathbf{i},\mathbf{j}}$ and let $w_{T,\mathbf{t}}$ be the basis elements defined similarly to [16]:

Let $w_{T,\mathbf{t}}$ be the sum of all $v_{\mathbf{i}}$ where \mathbf{i} is a multi-index such that for each column in $[\lambda]$ the numbers in \mathbf{t} in this column indicate the positions of the numbers in T in this column. To compute $\langle w_{T,\mathbf{t}}, v_{S,\beta} \rangle$ one has to find all multi-indices \mathbf{i} such that for each row (column) in $[\lambda]$ the numbers in β (\mathbf{t}) in this row (column) indicate the positions of the numbers in S (T) in this row (column). Each such multi-index contributes 1 or -1 to $\langle w_{T,\mathbf{t}}, v_{S,\beta} \rangle$. Let \mathbf{I} be the set of multi-indices satisfying these conditions.

Suppose first that $S = T$ and $\beta = \mathbf{t}$. Let k be the largest entry of T and let j be maximal such that k appears in column j in T . Suppose the entries equal to k in this column are in row $i+1, i+2, \dots, i+l$. The position of the l entries equal to k in $\mathbf{i} \in \mathbf{I}$ are numbers occurring in column j of \mathbf{t} . Since no entry in the first i rows of T is equal to k , the numbers in the first i rows of \mathbf{t} indicate positions with entries not equal to k . This uniquely determines the position of these l entries in \mathbf{i} . Repeating this process shows that there is a unique multi-index $\mathbf{i} \in \mathbf{I}$ which shows that $\langle w_{T,\mathbf{t}}, v_{T,\mathbf{t}} \rangle = \pm 1$ is invertible.

Suppose now that $\mathbf{t} \not\supseteq \beta$ and there exists a multi-index $\mathbf{i} \in \mathbf{I}$. By [12, 17] there are two entries, say k and l in some column i of \mathbf{t} which are in the same row j of β . Since \mathbf{i} satisfies the conditions above, i_k and i_l are in the i -th column of T and in the j -th row of S . In particular, $i_k \neq i_l$ and $\mathbf{i}(kl) \in \mathbf{I}$. Since $\mathbf{i}(kl)$ contributes -1 times the summand that is contributed by \mathbf{i} to $\langle w_{T,\mathbf{t}}, v_{S,\beta} \rangle$, we have $\langle w_{T,\mathbf{t}}, v_{S,\beta} \rangle = 0$.

Suppose now that $T \not\supseteq S$ and there is a multi-index $\mathbf{i} \in \mathbf{I}$. Since $T \not\supseteq S$ there is some i such that $\text{shape}(T \downarrow i) \not\supseteq \text{shape}(S \downarrow i)$. Delete all entries in β and in \mathbf{t} that indicate positions in \mathbf{i} of entries greater than i . Then each row of the tableau obtained from β has as many boxes as the corresponding row in S has, a similar statement holds for columns, \mathbf{t} and T . These new tableaux might have holes, but after pushing together entries in each row (column) of the tableau obtained from β (\mathbf{t}), these tableaux have the same shape as $S \downarrow i$

$(T \downarrow i)$. Again by the results in [12, 17] there are two entries in some column i of \mathbf{t} which are in the same row j of β . Now $\langle w_{T\mathbf{t}}, v_{S\beta} \rangle = 0$ follows as before. \square

8. A filtration of the mixed tensor space

Fix a natural number n . In this section, we define a basis of the mixed tensor space which leads to a filtration of the mixed tensor space with U - $B_{r,s}(n)$ -bimodules. Each layer of this filtration is a tensor product of a U -left module and a $B_{r,s}(n)$ -right module.

Recall the cellular basis of $E := \text{End}_U(V^{\otimes r} \otimes V^{*\otimes s})$

$$\{\overline{m}_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda, \mu), (\lambda, \mu) \in \Lambda_0(r, s)\}$$

from Theorem 32.

Definition 38. For $(\lambda, \mu) \in \Lambda_0(r, s)$ let $\overline{C}^{(\lambda, \mu)}$ be the cell module of E with R -basis $\{c_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda, \mu)\}$. By inflation, it is also a $B_{r,s}(n)$ -right module with action given by

$$c_{(\mathbf{t}, \mathbf{u}, \mathbf{v})}a = \sum_{(\mathbf{t}', \mathbf{u}', \mathbf{v}')} \lambda_{(\mathbf{t}', \mathbf{u}', \mathbf{v}')} c_{(\mathbf{t}', \mathbf{u}', \mathbf{v}')} a$$

for $a \in B_{r,s}(n)$ and $\lambda_{(\mathbf{t}', \mathbf{u}', \mathbf{v}')} \in R$ determined by

$$\overline{m}_{(\mathbf{t}', \mathbf{u}', \mathbf{v}'), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \overline{a} \equiv \sum_{(\mathbf{t}'', \mathbf{u}'', \mathbf{v}'')} \lambda_{(\mathbf{t}'', \mathbf{u}'', \mathbf{v}'')} \overline{m}_{(\mathbf{t}'', \mathbf{u}'', \mathbf{v}''), (\mathbf{t}', \mathbf{u}', \mathbf{v}')} \pmod{E(\triangleright(\lambda, \mu))},$$

where \overline{a} is the coset of a modulo ann and thus an element of E .

Definition 39. Let $0 \leq k \leq \min(r, s)$ and let $\lambda \in \Lambda(r-k)$ and $\mu \in \Lambda(s-k)$ be partitions with $\lambda_1 + \mu_1 \leq n$. A rational (λ, μ) -tableau is a pair $(\mathbf{a}, \mathbf{b}) \in \mathcal{T}(\lambda) \times \mathcal{T}(\mu)$.

Let $\text{first}_i(\mathbf{a}, \mathbf{b})$ be the number of entries of the first row of \mathbf{a} which are $\leq i$ plus the number of entries of the first row of \mathbf{b} which are $\leq i$. A rational tableau is called standard if $\mathbf{a} \in \mathcal{T}_0(\lambda)$, $\mathbf{b} \in \mathcal{T}_0(\mu)$ and the following condition holds:

$$\text{first}_i(\mathbf{a}, \mathbf{b}) \leq i \text{ for all } i = 1, \dots, n \quad (5)$$

We denote the set of standard rational (λ, μ) -tableaux by $\text{Rat}(\lambda, \mu)$.

Remark 40. Let (λ, μ) , $\lambda = (\lambda_1, \dots, \lambda_{l_\lambda})$, $\mu = (\mu_1, \dots, \mu_{l_\mu})$ be as in Definition 39 and let $l \geq l_\mu$ be a natural number. Let

$$\tau = (n^{l-l_\mu}, n - \mu_{l_\mu}, \dots, n - \mu_1, \lambda_1, \dots, \lambda_{l_\lambda}).$$

Then $\mathcal{T}_0(\tau)$ is in bijection with the set of standard rational (λ, μ) -tableaux (see [5]). By [19], the cardinality of this set is equal to the dimension of $V_{\lambda, \mu}$ in Proposition 14.

Definition 41. Let $(\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda, \mu)$ and $(\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M(\lambda, \mu)$. Let a_1, a_2, \dots, a_{r-k} and b_1, \dots, b_{s-k} be the entries of \mathbf{a} and \mathbf{b} respectively read row by row. Then $v_{\mathbf{a}, \mathbf{b}} := v_{a_{r-k}} \otimes \dots \otimes v_{a_2} \otimes v_{a_1} \otimes v_{b_1}^* \otimes v_{b_2}^* \otimes \dots \otimes v_{b_{s-k}}^*$ is an element of $V^{\otimes r-k} \otimes V^{*\otimes s-k}$. Set

$$v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})} := v_{\mathbf{a}, \mathbf{b}} m_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \in V^{\otimes r} \otimes V^{*\otimes s}$$

Theorem 42. The set

$$\{v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda, \mu), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda, \mu), (\lambda, \mu) \in \Lambda_0(r, s)\}$$

is an R -basis of mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$.

Proof. Since the rank of $V_{\lambda, \mu}$ in Proposition 14 for $(\lambda, \mu) \in \Lambda_0(r, s)$ is equal to the cardinality of $\text{Rat}(\lambda, \mu)$ and the multiplicity of $V_{\lambda, \mu}$ in $V^{\otimes r} \otimes V^{*\otimes s}$ is the cardinality of $M_0(\lambda, \mu)$, the cardinality of the set $\{v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}\}$ is equal to the rank of the mixed tensor space $V^{\otimes r} \otimes V^{*\otimes s}$. Thus it suffices to show that the set $\{v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}\}$ generates $V^{\otimes r} \otimes V^{*\otimes s}$.

The elements $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ can be defined even if (\mathbf{a}, \mathbf{b}) and $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ are not standard. It is easy to see that the $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ with all tableaux row-standard/row-anti-standard generate $V^{\otimes r} \otimes V^{*\otimes s}$. So we have to show that if (\mathbf{a}, \mathbf{b}) or $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ is not standard, then $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ can be written as a linear combination of such elements involving tableaux which are dominating in some sense.

Let $(\mathbf{a}, \mathbf{b}), (\mathbf{a}', \mathbf{b}') \in \text{Rat}(\lambda, \mu)$ and $(\mathbf{t}, \mathbf{u}, \mathbf{v}), (\mathbf{t}', \mathbf{u}', \mathbf{v}') \in M_0(\lambda, \mu)$. We define $((\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})) \supseteq ((\mathbf{a}', \mathbf{b}'), (\mathbf{t}', \mathbf{u}', \mathbf{v}'))$ if and only if $(\lambda, \mu) \triangleright (\lambda', \mu')$ or $(\lambda, \mu) = (\lambda', \mu')$, $\mathbf{a} \supseteq \mathbf{a}'$, $\mathbf{b} \supseteq \mathbf{b}'$ and $(\mathbf{t}, \mathbf{u}, \mathbf{v}) \supseteq (\mathbf{t}', \mathbf{u}', \mathbf{v}')$.

Now if $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ is not standard, by the previous results $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ can be written as a linear combination of dominating elements with respect to the order just defined. Suppose that \mathbf{a} is not semi-standard. Let $v_{\mathbf{a}, \mathbf{t}^\lambda}$ be the basis element as in Section 7. Then $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ involves a modified version $v_{\mathbf{a}, \mathbf{t}^\lambda}^{\text{refl}}$ of

this basis element with reversed ordering of the tensor product, i. e. it can be written as $(v_{\mathbf{a}, \mathbf{t}}^{\text{refl}} \otimes \dots) \cdot \dots$. Using Theorem 37, $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ can be written as a linear combination of dominating elements in the above sense. The same works if \mathbf{b} is not semi-standard.

Finally, suppose that Condition (5) does not hold and i is the minimal entry violating this condition. We first assume, that \mathbf{a} and \mathbf{b} are tableaux with one row and i is the greatest entry in \mathbf{a} and \mathbf{b} . Let $I = \{i_1, i_2, \dots, i_l\}$ where $i_1 < i_2 < \dots < i_l = i$ are the entries appearing both in \mathbf{a} and \mathbf{b} . Let $D = I \cup \{i+1, i+2, \dots, n\}$. Note that each element $v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}$ can be written as $v_{\mathbf{a}, \mathbf{b}} m_{(r-k), (s-k)} b$ for some linear combination of generalized diagrams b . Let M be the set of all rational $((r-k), (s-k))$ -tableaux $(\mathbf{a}', \mathbf{b}')$ such that \mathbf{a}' and \mathbf{b}' are row-standard and \mathbf{a}' and \mathbf{b}' are obtained from \mathbf{a} and \mathbf{b} by replacing the entries in I by entries in D . In particular $(\mathbf{a}, \mathbf{b}) \in M$ and all other elements of M satisfy Condition (5). By similar methods as in [5] one can show that $\sum_{(\mathbf{a}', \mathbf{b}') \in M} v_{\mathbf{a}', \mathbf{b}'} m_{(r-k), (s-k)}$ involves only dominating elements (such that the partitions involve less boxes). In the general case let i be again minimal violating the condition and let l_1 and l_2 be the number of entries $\leq i$ in the first row of \mathbf{a} and \mathbf{b} respectively. Then $m_{\lambda, \mu}$ has a left factor $m_{(l_1), (l_2)}$. Now, the result follows by plugging in the results for the special case and induction on the sum of all entries in the first rows. \square

The proof also shows the following proposition:

Corollary 43. *Let $(\lambda, \mu) \in \Lambda_0(r, s)$. Let $V(\supseteq (\lambda, \mu))$ be the R -span of the set*

$$\{v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda', \mu'), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda', \mu'), (\lambda', \mu') \supseteq (\lambda, \mu)\}$$

and similarly, let $V(\supset (\lambda, \mu))$ be the R -span of the set

$$\{v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda', \mu'), (\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda', \mu'), (\lambda', \mu') \supset (\lambda, \mu)\}.$$

Then $V(\supseteq (\lambda, \mu))$ and $V(\supset (\lambda, \mu))$ are U - $B_{r,s}(n)$ -submodules.

It is also easy to see that for a fixed triple $(\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda, \mu)$,

$$\langle v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda, \mu) \rangle_R + V(\supset (\lambda, \mu))$$

is a U -submodule and thus the following definition makes sense and is independent of the chosen triple $(\mathbf{t}, \mathbf{u}, \mathbf{v})$.

Definition 44. For $(\lambda, \mu) \in \Lambda_0(r, s)$ choose a triple $(\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda, \mu)$. Let $X^{(\lambda, \mu)}$ be the U -module

$$\langle v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})} \mid (\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda, \mu) \rangle_R + V(\triangleright(\lambda, \mu)) / V(\triangleright(\lambda, \mu)).$$

We will show that $X^{(\lambda, \mu)}$ are dual Weyl modules. Let D^+ and D^- respectively be the one-dimensional U -modules with basis $\{d^+\}$ and $\{d^-\}$ on which the generators $E_i = \theta_{i+1}^i$ and $F_i = \theta_i^{i+1}$ act as zero, $\theta_i^i d^+ = d^+$ and $\theta_i^i d^- = -d^-$. Let $S : U \rightarrow U$ be the antipode of U , i. e. the antiisomorphism fixing E_i and F_i and mapping θ_i^i to $-\theta_i^i$. If M is a U -module, then M^* becomes a U -module setting $uf(m) = f(S(u)m)$.

Now, $\iota : V^* \rightarrow V^{\otimes n-1} \otimes D^- : v_i \mapsto (-1)^i v_{(12\dots\hat{i}\dots n)} y_{(n-1)} \otimes d^-$ defines a U -monomorphism (\hat{i} means omitting i). Likewise, one can define a U -monomorphism $\iota : V^{\otimes r} \otimes V^{*\otimes s} \rightarrow V^{\otimes r+(n-1)s} \otimes D^{-\otimes s}$. If $\{1, \dots, n\} = \{i_1, \dots, i_l\} \dot{\cup} \{j_1, \dots, j_{n-l}\}$ is a disjoint union, then it can be seen easily, that

$$\iota((v_{i_1}^* \otimes \dots \otimes v_{i_l}^*) y_{(l)}) = \pm (v_{j_{n-l}} \otimes \dots \otimes v_{j_1}) y_{(n-l)} \otimes (v_{(1\dots n)} y_{(n)})^{\otimes l-1} \otimes d^{-\otimes l}.$$

Furthermore, $\iota(\sum_{i=1}^n v_i \otimes v_i^*) = v_{(1\dots n)} y_{(n)}$.

If $(\mathbf{a}, \mathbf{b}) \in \text{Rat}(\lambda, \mu)$, let τ be the partition from Remark 40 and \mathbf{c} be the standard τ -tableau in bijection with (\mathbf{a}, \mathbf{b}) . Then for each $(\mathbf{t}, \mathbf{u}, \mathbf{v}) \in M_0(\lambda, \mu)$ there exists a permutation $\pi_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \in \mathfrak{S}_{r+(n-1)s}$ such that $\iota(v_{(\mathbf{a}, \mathbf{b}), (\mathbf{t}, \mathbf{u}, \mathbf{v})}) = \pm v_{\mathbf{c}, \mathbf{t}^\tau} \pi_{(\mathbf{t}, \mathbf{u}, \mathbf{v})} \otimes d^{-\otimes s}$. This shows that $\iota(V(\triangleright(\lambda, \mu))) \subseteq V(\triangleright\tau) \otimes D^{-\otimes s}$ and $X^{(\lambda, \mu)} \cong X^\tau \otimes D^{-\otimes s}$ with $X^\tau = X^{(\tau, \emptyset)}$.

Let τ^t be the transpose of τ . By inspecting the action of U on X^τ , one can see that X^τ is isomorphic to the module $D_{\tau^t, R}$ defined in [9]. Note that the modules in [9] can be viewed as modules over U , since the Schur algebras are epimorphic images of U . Furthermore, $D_{\tau^t, R} \cong V_{\tau^t, R}^\circ$ (notation as in [9]) is isomorphic to ${}^\omega \Lambda_{\tau^t}^* \cong \Lambda_{-w_0 \tau^t}^*$ with ω , Λ_\square and w_0 defined as in [15].

Proposition 45. Let $\lambda^t = (\lambda'_1, \lambda'_2, \dots, \lambda'_l)$ and $\mu^t = (\mu_1, \dots, \mu'_l)$. Then we have

$$X^{(\lambda, \mu)} \cong \Lambda_{(\mu'_1, \mu'_2, \dots, \mu'_l, 0, \dots, 0, -\lambda'_1, \dots, -\lambda'_2, -\lambda'_1)}^*.$$

Proof. $X^{(\lambda, \mu)} \cong \Lambda_{-w_0 \tau^t}^* \otimes D^{-\otimes s} \cong \Lambda_{-w_0 \tau^t}^* \otimes D^{+\otimes s} \cong (\Lambda_{-w_0 \tau^t} \otimes D^{+\otimes s})^* \cong \Lambda_{-w_0 \tau^t + s \cdot (1^n)}^*$ and the claim follows by computing $-w_0 \tau^t + s \cdot (1^n)$. \square

The modules Λ_\square are often (but not always) called Weyl modules, thus Proposition 45 shows that the modules $X^{(\lambda, \mu)}$ are dual Weyl modules where dual modules are defined via the antipode.

Theorem 46 (Filtration with cell modules and dual Weyl modules). *We have*

$$V(\supseteq (\lambda, \mu))/V(\supset (\lambda, \mu)) \cong X^{(\lambda, \mu)} \otimes \overline{C}^{(\lambda, \mu)}$$

as $U\text{-}B_{r,s}(n)$ -bimodules. In particular, $V^{\otimes r} \otimes V^{\otimes s}$ has a filtration with cell modules and one with dual Weyl modules.*

Proof. This is straight forward from the construction of the basis. \square

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